Invariant surfaces in a 3-manifold with constant (mean or Gaussian) curvature

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Invariant surfaces in a 3-dimensional manifold

Let $(N^3, g)$ be 3-dimensional Riemannian manifold and let $X$ be a **Killing** vector field on $N$.

$X$ generates a **one-parameter subgroup** $G_X$ of the Lie group of **isometries** of $(N^3, g)$ which acts on $N$ by isometries.

An **orbit** $G(p)$, $p \in N$, is called **principal** if there exists an open neighbourhood $U \subset N$ of $p$ such that all orbits $G(q)$, $q \in U$, are diffeomorphic to $G(p)$.

Let $N_r = \{ p \in N : G(p) \text{ principal} \}$ be the **regular** part of $N$

If $N/G$ is connected, from the **Principal Orbit Theorem** we have

- the principal orbits are all diffeomorphic

- $N_r$ is open and dense in $N$

- the orbit space $N_r/G_X$ is a connected manifold and
  \[
  \pi : N_r \to N_r/G_X \quad \text{is a submersion}
  \]
**Definition.** An immersion
\[ \varphi : M^2 \to (N^3_r, g) \]
is \(G_X\)-equivariant if there exists an action of \(G_X\) on \(M\) such that
\[ \varphi(gx) = g\varphi(x) \]

A \(G_X\)-equivariant immersion \(\varphi : M^2 \to (N^3, g)\) induces on \(M^2\) a Riemannian metric, the pull-back metric, denoted by \(g\varphi\) and called the \(G_X\)-invariant induced metric.

**Problem:** Describe the \(G_X\)-invariant immersions such that:

- The Gauss curvature \(K = \text{cst}\)
- The mean curvature \(H = \text{cst}\)
Reduction techniques

Let $\varphi : M^2 \to N_r$ be a $G_X$-equivariant immersion then we have the following diagram:

\[(M^2, g_f) \xrightarrow{\varphi} (N^3, g) \quad \pi \quad \leftarrow \text{Riem. Sub.} \]

\[M^2/G_X \xrightarrow{\tilde{\varphi}} (N^3_r/G_X, \tilde{g})\]

The quotient (orbit) metric $\tilde{g}$ is described as follows:

the orbit space $N_r/G_X$ can be locally parametrized by the invariant functions of the Killing vector field $X$ ($f$ invariant iff $Xf = 0$).

If \( \{f_1, f_2\} \) is a complete set of invariant functions then

\[
\tilde{g} = \sum_{i,j=1}^{2} h^{ij} df_i \otimes df_j
\]

where \((h^{ij})\) is the inverse of the matrix

\[
h_{ij} = g(\nabla f_i, \nabla f_j)
\]
• $\tilde{\gamma} : (a, b) \subset \mathbb{R} \rightarrow (N_r^3 / G_X, \tilde{g}), \quad \|\tilde{\gamma}'\| = 1$

  a curve on the orbit space

• $\gamma : (a, b) \subset \mathbb{R} \rightarrow N_r^3$

  a lift of $\tilde{\gamma}$ such that $d\pi(\gamma') = \tilde{\gamma}'$.

• $\omega = \|X(\gamma(t))\|_g$

• $N$ a unit normal vector to the image of $M$ in $N$


**Theorem:** (Reduction Theorem for $H$).

\[ H = k_{\tilde{\gamma}} - g(\text{grad} \ln \omega, N) \]

where $k_{\tilde{\gamma}}$ is the geodesic curvature of $\tilde{\gamma}$ in the orbit space.
For the Gauss curvature we have

**Theorem:** (I.I. Onnis, M. (2004)).

(i) If the $G_X$-invariant induced metric $g_\varphi$ is of **constant** Gaussian curvature $K$, then the function $\omega(t) = \|X(\gamma(t))\|_g$ satisfies the following ODE

$$\frac{d^2}{dt^2} \omega(t) + K \omega(t) = 0.$$  \hspace{1cm} (1)

(ii) Vice versa, suppose that Equation (1) holds with $K$ a real constant. Then, in all points where $d(\omega^2)/dt \neq 0$, the $G_X$-invariant induced metric $g_\varphi$ has constant Gauss curvature.
Rotational surfaces in $\mathbb{R}^3$ with constant Gaussian curvature

- $K < 0$ (Minding 1839)

Hyperbolic type  Conic type  Pseudosphere

- $K > 0$ (Enneper 1868)

Spindle type  Barrel type
Rotational surfaces in $\mathbb{R}^3$ with constant mean curvature

Delaunay’s surfaces 1841

- $H = 0$
  - Plane
  - Catenoid

- $H \neq 0$

Kenmotsu’s formula 1980

\[
\alpha[H, B](s) = \left\{ \begin{array}{l}
\int_{0}^{s} \frac{1 + B \sin(2Ht)}{\sqrt{1 + B^2 + 2B \sin(2Ht)}} dt \\
\sqrt{1 + B^2 + 2B \sin(2Hs)}
\end{array} \right.
\]
$B = 0$ Straight line parallel to the rotational axis.

$B = 1$ Sequence of spheres with the same radius

$B > 1$ Nodary curve

$B < 1$ Unduloid curve
Helicoidal surfaces in $\mathbb{R}^3$ with constant mean curvature

Theorem: (Dajczer and Do Carmo (1982)).

Helicoidal surfaces of constant mean curvature $H$ can be isometrically deformed to rotational surfaces preserving the mean curvature.

- $H = 0$ From Catenoid to Helicoid
Helicoidal surfaces in $\mathbb{R}^3$ with constant mean curvature of nodary type
Helicoidal surfaces in $\mathbb{R}^3$ with constant Gauss curvature
From the Pseudosphere to the surface of U. Dini
Helicoidal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature

Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|^2 < 2\}$ be the disk model of the hyperbolic plane and consider $\mathbb{H}^2 \times \mathbb{R}$ endowed with the metric

$$ds^2 = \frac{dx^2 + dy^2}{F^2} + dz^2$$

where $F = \frac{2-x^2-y^2}{2}$.

The Lie algebra of the infinitesimal isometries of $(\mathbb{H}^2 \times \mathbb{R}, ds^2)$ admits the following bases of Killing vector fields

$$X_1 = (F + y^2) \frac{\partial}{\partial x} - xy \frac{\partial}{\partial y}$$

$$X_2 = -xy \frac{\partial}{\partial x} + (F + x^2) \frac{\partial}{\partial y}$$

$$X_3 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

$$X_4 = \frac{\partial}{\partial z}$$

A one-dimensional subgroup $G$ of $Isom(\mathbb{H}^2 \times \mathbb{R})$ is called helicoidal if it is generated by linear combinations of

$$bX_3 + aX_4 \quad a, b \in \mathbb{R}.$$ 

If $b = 0$ the group is translational, while if $a = 0$ the group is rotational.
Classification of helicoidal surfaces

In cylindrical coordinates \((r, \theta, z)\) in \(\mathbb{H}^2 \times \mathbb{R}\)

\[
X_3 + aX_4 = \frac{\partial}{\partial \theta} + a \frac{\partial}{\partial z}
\]

the invariant functions are

\[
u = f_1 = r \quad v = f_2 = z - a\theta,
\]

and the orbit space is

\[
\mathcal{B} = \{(u,v) \in \mathbb{R}^2 : 0 \leq u < \sqrt{2}\}
\]

with the orbital metric

\[
\tilde{g} = \frac{du^2}{F^2} + \frac{r^2}{r^2 + F^2a^2} dv^2.
\]

The **Reduction-Theorem** implies that a curve \(\gamma(s) = (u(s), v(s))\) in the orbit space generates a CMC surface iff

\[
\begin{aligned}
\dot{u} &= F \cos \sigma \\
\dot{v} &= \frac{(u^2 + F^2a^2)}{u} \sin \sigma \\
\dot{\sigma} &= H - \frac{(u^2 + 2)}{2u} \sin \sigma
\end{aligned}
\]

(2)

A prime integral of (2) is

\[
J(s) = \frac{u \sin \sigma - H}{u^2 - 2} = k = \text{const}.
\]
Thus, the profiles of the helicoidal CMC surfaces are solutions of the equation

\[
\frac{u \sin \sigma - H}{u^2 - 2} = k,
\]

for all \( k \in \mathbb{R} \).

Setting \( C' = k - H/2 \), we have

**Theorem: (Hsiang,Hsiang - I.I. Onnis, M.).** Let \( \Sigma \subset \mathbb{H}^2 \times \mathbb{R} \) be a CMC helicoidal surface and let \( \gamma = \Sigma / G \) be the profile curve in the orbit space. Then we have the following characterization of \( \gamma \) according to the value of \( H \).

I) \( (H > \sqrt{2}) \) - The profile curve is of Delaunay type. Moreover if

- \( C > 0 \) is of **nodary-type**
- \( C = 0 \) is of **circle-type**
- \( C < 0 \) is of **undulary-type** or a vertical straight line

II) \( (H = \sqrt{2}) \) - The profile curve is

- for \( C > 0 \) of **folium-type**
- for \( C = 0 \) of **conic-type**
- for \( C < 0 \) of **bell-type**

III) \( (H < \sqrt{2}) \) - The profile curve is

- for \( C > 0 \) of **bounded folium-type**
- for \( C = 0 \) of **helicoidal-type** or a horizontal straight line for \( H = 0 \)
- for \( C < 0 \) of **catenary-type**.
- Profile curves for $H > \sqrt{2}$

- Profile curves for $H = \sqrt{2}$

- Profile curves for $H < \sqrt{2}$
Translational surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature

In this case the Reduction-Theorem gives that a curve $\gamma(s) = (u(s), v(s))$ in the orbit space $B = \mathbb{H}^2$ generates a translational surface if $u$ and $v$ satisfy the following system

$$\begin{align*}
\dot{u} &= F \cos \sigma \\
\dot{v} &= F \sin \sigma \\
\dot{\sigma} &= H - u \sin \sigma + v \cos \sigma.
\end{align*}$$

The prime integral is

$$J(s) = \frac{\dot{\sigma}}{2F} = k = \text{cst}$$

**Theorem:** The CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant under the action of the subgroup $G$ generated by the Killing vector field $X_4 = \frac{\partial}{\partial z}$ are:

- $H = 0$
  - (a) $k = 0$ part of a **vertical plane** through the origin;
  - (b) $k \neq 0$ part of a **right cylinder** with directrix a **geodesic circle** on the hyperbolic disk.

- $H > 0$
  - (a) $k = 0$ part of **vertical plane** not through the origin;
  - (b) $k \neq 0$ part of a **right cylinder** with diretrix a **non** geodesic circle on the hyperbolic disk.
The Heisenberg space

The 3-dimensional **Heisenberg** group $\mathbb{H}_3$ is the two-step nilpotent Lie group standardly represented in $Gl_3(\mathbb{R})$ by

$$
\begin{bmatrix}
1 & x_1 & x_3 + \frac{1}{2}x_1x_2 \\
0 & 1 & x_2 \\
0 & 0 & 1
\end{bmatrix}
\quad x_i \in \mathbb{R}
$$

Endowed with the **left-invariant** metric

$$g = dx_1^2 + dx_2^2 + \left(dx_3 + \frac{1}{2}x_2dx_1 - \frac{1}{2}x_1dx_2\right)^2$$

the isometry group of $(\mathbb{H}_3, g)$ has dimension 4 and a basis of Killing vector fields is given by:

$$
\begin{align*}
X_1 &= \frac{\partial}{\partial x_1} + \frac{y_2}{2} \frac{\partial}{\partial x_3} \\
X_2 &= \frac{\partial}{\partial x_2} - \frac{x_1}{2} \frac{\partial}{\partial x_3} \\
X_3 &= \frac{\partial}{\partial x_3} \\
X_4 &= -y_2 \frac{\partial}{\partial x_3} + y_1 \frac{\partial}{\partial x_2}
\end{align*}
$$
The 1-dimensional subgroups of the isometry group $Isom(\mathbb{H}_3, ds^2)$ can be divided in two families:

1) the one parameter subgroups generated by linear combinations

\[ a_1 X_1 + a_2 X_2 + a_3 X_3 + bX_4, \]

with $b \neq 0$. These subgroups are called of helicoidal type. If $a_i = 0$ for $i \in \{1, 2, 3\}$, we obtain the group $SO(2)$ generated by $X_4$;

2) the one parameter subgroups generated by linear combinations of $X_1$, $X_2$ and $X_3$, called of translational type.

**Lemma: (Figueroa-Mercuri-Pedrosa).** A surface in $\mathbb{H}_3$ which is invariant under the action of a one-parameter subgroup of isometries $G_X$ generated by a Killing vector field $X = \sum_i a_i X_i$ is isometric to a surface invariant under the action of one of the following groups

\[ G_1, \quad G_3, \quad G_{34}. \]
Helicoidal surfaces in $\mathbb{H}_3$ with constant mean curvature

In this case the function

$$J(s) = u \sin \sigma - \frac{1}{2} Hu^2$$

is constant along a profile curve of a helicoidal CMC surfaces.

Then, studying the equation $J(s) = k$, we have

(Tompter ’93, Caddeo-Piu-Ratto ’94, Figueroa-Mercuri-Pedrosa ’99)

**Theorem:** The helicoidal CMC surfaces of $\mathbb{H}_3$ are, in terms of $H$ and $k$:

- $H = 0$ (minimal surfaces)
  
  (a) $k = 0$ **helicoids**, including horizontal planes

  (b) $k \neq 0$ surfaces generates by curves of the **catenaria** type

- $H > 0$
  
  (a) $k = 0$ compact surfaces of **spherical** type

  (b) $k \neq 0$ right cylinders and surface of **De-launay** type
Translational surfaces in $\mathbb{H}_3$ with constant mean curvature

**Theorem:** (Figueroa-Mercuri-Pedrosa). The CMC surfaces of $\mathbb{H}_3$ invariant under $G_1$ are:

- $H = 0$ The vertical planes

- $H > 0$ Vertical right cylinders with (Euclidean) radii $H^{-1}$

**Theorem:** (Figueroa-Mercuri-Pedrosa). The CMC surfaces of $\mathbb{H}_3$ invariant under $G_3$ are:

- $H = 0$

  (a) The surface of equation

  \[
  z = \frac{xy}{2} - c \left[ y \frac{1 + y^2}{2} + \frac{1}{2} \ln(y + \frac{1}{1 + y^2}) \right]
  \]

  (b) Vertical planes

- $H > 0$ The surface of equation

  \[
  z = \frac{xy}{2} \pm \frac{1}{2H} \left[ \frac{1 - H^2y^2}{1 + y^2 + \frac{1 + H^2}{H} \arcsin \sqrt{\frac{1 - H^2y^2}{1 + H^2}}} \right]
  \]
Invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with constant Gauss curvature

Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the half plane model of the hyperbolic plane and consider $\mathbb{H}^2 \times \mathbb{R}$ endowed with the product metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + dz^2.$$

The Lie algebra of the infinitesimal isometries of the product $(\mathbb{H}^2 \times \mathbb{R}, ds^2)$ admits the following bases of Killing vector fields

$$X_1 = \frac{(x^2 - y^2)}{2} \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$$

$$X_2 = \frac{\partial}{\partial x}$$

$$X_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

$$X_4 = \frac{\partial}{\partial z}.$$

**Lemma: (I.I. Onnis, M.).** Any $G_X$-invariant surface in $\mathbb{H}^2 \times \mathbb{R}$ is isometric to a surface invariant under the action of

$$G_{24}, \quad G_{34}, \quad G_{12}^*, \quad G_{124}^*,$$

where $G_{12}^*$ is generated by $X_{12}^* = X_1 + (X_2)/2$ and $G_{124}^*$ is generated by $X_{12}^*$ and $X_4$. 
**Theorem:** (I.I. Onnis, M.). Let $\tilde{\gamma} = (u(s), v(s))$ be a curve in the orbit space $(\mathbb{H}^2 \times \mathbb{R} / G, \tilde{g})$, parametrized by arc length, which is the profile curve of a $G$-invariant surface in $(\mathbb{H}^2 \times \mathbb{R})$. Then:

- if $G = G_4$, the orbit space is $\mathbb{H}^2$ and any curve is the profile curve of a flat $G_4$-invariant cylinder;

- if $G = G_{24}$, $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 : u > 0\}$ $\tilde{g} = \frac{du^2}{u^2} + \frac{dv^2}{a^2 + b^2 u^2}$ and the profile curve can be parametrized by

$$
\begin{aligned}
  u(s) &= |a|/\sqrt{\omega^2 - b^2}, \quad a, b \in \mathbb{R} \\
  v(s) &= \int_{s_0}^{s} \frac{a^2 \omega^2}{\omega^2 - b^2} \left[ 1 - \left( \frac{\omega' \omega}{\omega^2 - b^2} \right)^2 \right] dt
\end{aligned}
$$

- if $G = G_{34}$, $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 : 0 < u < \pi\}$ $\tilde{g} = \frac{du^2}{\sin^2(u)} + \frac{dv^2}{a^2 + b^2 \sin^2(u)}$ and the profile curve can be parametrized by

$$
\begin{aligned}
  u(s) &= \arcsin \left( \frac{|a|}{\sqrt{\omega^2 - b^2}} \right), \quad a, b \in \mathbb{R} \\
  v(s) &= \int_{s_0}^{s} \sqrt{\frac{a^2 \omega^2}{\omega^2 - b^2}} \left[ 1 - \frac{(\omega')^2}{(\omega^2 - b^2)(\omega^2 - a^2 - b^2)} \right] dt
\end{aligned}
$$

- if $G = G_{124}^*$, $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 : u \geq 2\}$ $\tilde{g} = \frac{du^2}{u^2 - 4} + \frac{(u^2 - 4) dv^2}{u^2 + 4(a^2 - 1)}$ and the profile curve can be parametrized by

$$
\begin{aligned}
  u(s) &= 2 \sqrt{\omega^2 + 1 - a^2}, \quad a \in \mathbb{R} \\
  v(s) &= \int_{s_0}^{s} \sqrt{\frac{\omega^2}{\omega^2 - a^2}} \left[ 1 - \frac{(\omega')^2}{(\omega^2 - a^2)(\omega^2 + 1 - a^2)} \right] dt
\end{aligned}
$$
**Theorem: (I.I. Onnis, M.).** Let $\gamma = (u(s), v(s))$ be a curve in the orbit space $(H_3/G, \tilde{g})$ parametrized by arc length which is the profile curve of a $G$-invariant surface in $H_3$ then:

- If $G = G_1$, $B = \mathbb{R}^2$ $\tilde{g} = du^2 + \frac{dv^2}{u^2+1}$ and the profile curve can be parametrized by
  \[
  \begin{aligned}
  u(s) &= \frac{\sqrt{\omega^2 - 1}}{
  \int_{s_0}^s \omega^2 \left[ 1 - \frac{(\omega \omega')^2}{(\omega^2 - 1)} \right] dt}
  \\
v(s) &= \int_{s_0}^s \frac{\omega^2}{1 - \frac{(\omega \omega')^2}{(\omega^2 - 1)}}
  \end{aligned}
  \]

- If $G = G_3$, $B = \mathbb{R}^2$ $\tilde{g} = du^2 + dv^2$ and, as $\omega = 1$, any curve parametrized by arc length is the profile curve of a flat $G_3$-invariant vertical cylinder.

- If $G = G_{34}$, $B = \{(u, v) \in \mathbb{R}^2 : u \geq 0\}$ $\tilde{g} = du^2 + \frac{4u^2 dv^2}{4u^2 + (u^2 + 2a)^2}$ and the profile curve can be parametrized by
  \[
  \begin{aligned}
  u(s) &= 2 \left( \sqrt{\omega^2 + 2a + 1 - a - 1} \right) \quad a \in \mathbb{R}^+
  \\
v(s) &= \omega \int_{s_0}^s \frac{2(\omega^2 + 2a + 1)^{\frac{3}{2}} - (2a + 1)\omega^2 - 4a^2 - 6a - 2 - \omega^2}{(\omega^2 + 2a + 1)(\sqrt{\omega^2 + 2a + 1 - a - 1})^2} dt
  \end{aligned}
  \]

  When $a = 0$ we have the $SO(2)$-invariant surfaces.

The case of $SO(2)$-invariant surfaces with constant Gauss curvature was described explicitly by Caddeo-Piu-Ratto.