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« Brownian motion and symplectic geometry »

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Dual role of $H_0 = -\frac{\hbar^2}{2}\Delta$ in ($\hbar$ = Planck's constant > 0)

Statistical Mechanics	Quantum Mechanics
Brownian motion	Free particle
(1) $\hbar\partial_t\eta = H_0\eta$	**(2)** $i\hbar\partial_t\Psi = H_0\Psi$ in L^2 (« Probabil. interpretation »)

Wiener measure $\mu_w^{\hbar}$ on $\Omega_0 = \left\{\omega \in C\left(IR^+, IR\right) \text{ t.q } \omega(0) = 0\right\}$

- Prob. distribution of $\omega(t)$ under $\mu_w^{\hbar}(d\omega)$:

$$\mu^{\hbar}\left(\omega(t) \in dq\right) = h_0(0,t,q)dq = (2\pi\,\hbar\, t)^{-1/2}\exp\left\{-\frac{1}{2\hbar t}q^2\right\}dq$$

- $\forall \ 0 = t_0 < t_1 < \ldots < t_n$,

$\omega(t_i) - \omega(t_{i-1}), \ i = 1,\ldots,n$ independent under $\mu^{\hbar}(d\omega)$

$(\Omega_0, \beta, \mu^{\hbar}) =$ Prob. space $\quad \{\omega(t) \equiv w_t\}_{t \in \mathrm{IR}^+} =$ Brownian motion

Intuitive definition (Feynman):

$L_0(\dot{\omega})$, free Lagrangian associated with H_0

(3) $$\mu^{\hbar}(d\omega) = C \ \exp\left(-\frac{1}{\hbar} \int_0^{\infty} \frac{1}{2} \left| \frac{d}{dt} \omega(t) \right|^2 dt \right) D\omega$$

constant $\quad$ Action $S_{L_0}[\omega]$

$D\omega$ = « Lebesgue measure » on Ω_0

$$t \rightarrow -\sqrt{-1}\, t$$

Brownian motion $\longleftrightarrow$ Quantum free particle

w_t simpler than quantum particle : $E[w_t] = 0$

constant « drift »

$$\begin{cases} \langle Q \rangle_{\psi_t} = \int q |\psi_t(q)|^2 dq = \alpha + \beta t \\ \langle P \rangle_{\psi_t} = \int -i\hbar \frac{\nabla \psi_t}{\psi_t} |\psi_t(q)|^2 dq = \beta \end{cases}$$

For good analogy, need of diffusions $z_t : I \rightarrow IR$ built from $(\prec\prec)$ Brownian w_t and Eq (1) exclusively, but with drift. $\langle Q \rangle_{\psi_t}, \langle P \rangle_{\psi_t}$ suggest

(4)

$$dz_t = \underbrace{\hbar^{1/2} dw_t}_{\text{Noise}} + \underbrace{\hbar \nabla \log \eta_t(z_t) dt}_{\text{ODE}}$$

$\frac{dw_t}{dt}$ only a distribution ! $\rightarrow$ differential notation

Sol z_t of (Itô's) SDE (4) is (time indexed) random variable.

$t \mapsto z_t$ not differentiable.

To each given solution $\eta_t > 0$ of (1) is associated a z_t.

Trivial example : $\eta_t = 1 \quad \forall t \rightarrow z_t = \hbar^{1/2} w_t$

(Itô's) differential & integral calculus for Eq (4) = $\hbar$-deformation of Leibniz-Newton calculus for smooth $t \mapsto q(t)$, built on properties of heat equation (1).

Fundamental property of $t \mapsto z_t$:

(5) $E_t\left[\left(z_{t+\Delta t} - z_t\right)^2\right] = \hbar \Delta t + (\Delta t)$

↑ conditional expectation

Ito's calculus : $dw_t^2 = dz_t^2 = \hbar \, dt$

$\Rightarrow$ Taylor expansion of $\forall$ $F(z_t)$ up to <u>second</u> order

Natural expansion to : $I \rightarrow M$ *n*-dim. Riemannian manifold

$t \mapsto z_t$

(6) $\hbar\partial_t\eta = H_0\eta \quad H_0 = -\frac{\hbar^2}{2}\nabla^j\nabla_j$ ⟵ covariant derivative w.r.t.

Riemannian connection,
Christoffel Γ^i_{jk}.

z_t provides Riemannian metric g^{ij} :

(7) $dz^j_t = dw^j_t + \left(\hbar\nabla^j \log\eta_t - \frac{\hbar}{2}\Gamma^k_{ik}g^{ik}\right) dt$

(8) $dw^i dw^j = dz^i dz^j = g^{ij}dt$. If $M = IR^n, g^{ij} = \delta^{ij}$, *n*-dim. $z_t = w_t$

There are interesting new (geometrical) features of the theory but no qualitatively new ideias.

Theory of SDE on Riemannian manifolds (« Stochastic Riemannian Geometry ») as old as in IR^n.

General Theory :

$$\underbrace{\text{« singular »}}_{\text{Ito's stochastic integral}} \quad \underbrace{\text{« smooth »}}$$

(9) $dz_t = \underbrace{\tilde{\sigma}_i(z_t)dw_t^i} + \tilde{B}(z_t)dt$

under restrictions on $\tilde{\sigma}, \tilde{B} \to$ differential and integral calculus on $\infty-$dimensional space, Ω_0, based on μ_w (Wiener, Cameron-Martin, Girsanov, Itô, Malliavin, …)

$\Omega_0 \supset \aleph \equiv$ Cameron – Martin Hilbert space :

$$\aleph = \left\{ q \in \Omega_0, \text{ absol. cont. and } \|q\|_1^2 = \int_0^\infty |\dot{q}(\tau)|^2 d\tau < \infty \right\}$$

shift $T_q : \Omega_0 \to \Omega$ induces $P_q \prec\prec \mu_w \quad \Leftrightarrow \quad q \in \aleph$.

Although $\mu_w(\aleph) = 0$, $\dfrac{dP_q}{d\mu_w}$ defines Ito's stochastic integral as

extension from ODE to SDE.

$\rightarrow$ Differential Geometry where $\aleph =$ tangent space of Ω_0

Intuitive def. of $\mu^{\hbar} \rightarrow$ Technical difficulties :

Typical $\omega \in \Omega_0 \rightarrow$ Action $S_{L_0}[\omega] = \infty$: Apparent singularities !

Very reminiscent of Feynman's path integral approach to quantum theory !

Kinematical property of quantum paths $t \rightarrow \omega(t)$

$$\left\langle \left(\omega(t+\Delta t) - \omega(t)\right)^2 \right\rangle_{S_L} = i\hbar\Delta t$$

where

Lagrangian

(10) $S_L[q] = \int L(\dot{q}(t), q(t))\, dt$

$$\equiv \int \omega_{pc} \quad , \qquad \omega_{pc} = pdq - hdt$$

Poincaré-Cartan form

(p = momentum, h=h(q,p) Hamiltonian) along (smooth !) solutions $t \mapsto q(t)$ of classical Euler-Lagrange equations.

$\langle ... \rangle_{S_L}$ = Feynman's path integral reinterpretation of quantum expectation (in terms of spectral measure of self-adjoint operators in L^2)

Same coexistence, here, of smooth (classical) and singular (diveregent) expressions.

But Probability Measures provide tools to (partly) eliminate such divergences :

$dz_t = \tilde{\sigma}_i(z_t)dw_t^i + \tilde{B}(z_t)dt \Rightarrow \dfrac{dz_t}{dt}$ meaningless, but

(11) $\lim_{\Delta t \downarrow 0} E_t \left[\dfrac{z_{t+\Delta t} - z_t}{\Delta t} \right] \equiv Dz_t$ makes sense !

For such a z_t, $D = \dfrac{\partial}{\partial t} + \tilde{B}\dfrac{\partial}{\partial q} + \dfrac{\hbar}{2}\dfrac{\partial^2}{\partial q^2}$ on

any regular $F = F(q,t)$, for $q = z_t$. Ex: $F = q$, $D_t z_t = \tilde{B}(z_t)$.

Under E_t, D behaves as derivative :

$$\forall\, F \in \mathcal{D}_D \qquad \underbrace{E_{q,t}\int_t^T DF(z_\tau,\tau)\, d\tau}_{\text{Riemann-Stieltjes !}} = E_{q,t}F(z_T,T) - F(q,t)$$

Example :

$$D\big(F(z_t)G(z_t)\big) = FDG + DF.G + \hbar\nabla F.\nabla G$$

Idea : Use Dz_t as variable <u>independent</u> of z_t like in classical Lagrangian dynamics.

Problem : Given SDE for z_t, (z_t, Dz_t) not independent !

Related problem : Lack of « <u>Stochastic Symplectic Geometry</u> »

How to built natural Liouville measure (or 2 form $\Omega = dp \wedge dq$) in Stochastic Analysis , when $q = z_t$ as before ?
Not a single result !

Quantum theory should help, where roles of position (z_t) and momentum (Dz_t) are sharply distinguished.

Our approach : Use Cartan's geometry of heat eq(1) ($\equiv$ quantum deformation of classical dynamics) and read results on $q = z_t$.

ω_{pc} and drift of SDE (3) suggests $\tilde{S} \equiv -\hbar \ell n\, \eta_t(q)$ $(\eta_t > 0$ solving (1))

(12) $$\frac{\partial \tilde{S}}{\partial t} = \frac{1}{2}\left(\frac{\partial \tilde{S}}{\partial q}\right)^2 - \frac{\hbar}{2}\frac{\partial^2 \tilde{S}}{\partial q^2} \qquad \text{(free) HJB}$$

Def : Energy $E = -\frac{\partial \tilde{S}}{\partial t}$, $B = -\frac{\partial \tilde{S}}{\partial q}$ momentum

$\forall$ solution of HJB annuls (classical) differential forms on
$M = IR^5(q,t,B,E,S)$:

(i.e. d $\equiv$ exterior derivative !)

(13)

$$\Omega \equiv \begin{cases} \omega = dS + Bdq + Edt & \text{(contact form)} \\ d\omega = dBdq + dEdt & \text{(Liouville in extended phase space)} \\ \beta = \left(E + \frac{1}{2}B^2\right)dqdt + \frac{\hbar}{2}dBdt & \text{(HJB . dqdt)} \end{cases}$$

$(\omega, d\omega, \beta)$ = Differential ideal I_{HJB} of (12) $\left(d\beta = (-dq + Bdt)d\omega\right)$

« Isovector » $N = N^q \frac{\partial}{\partial q} + N^t \frac{\partial}{\partial t} + N^B \frac{\partial}{\partial B} + N^E \frac{\partial}{\partial E} + N^S \frac{\partial}{\partial S}$ of I_{HJB} :

(14) $L_N(I_{HJB}) \subset I_{HJB}$

Lie derivative w.r.t. N

{N} = Lie algebra $\mathfrak{g}$ of symmetry group of HJB.

To compute base of $\mathfrak{g}$, solve system of linear PDE for components N^q, N^t, … of N.

Result : (for $q \in IR$)

$$N_1 = \frac{\partial}{\partial t}, \qquad N_2 = \frac{\partial}{\partial q}, \qquad N_3 = -\hbar \frac{\partial}{\partial S}$$

$$N_4 = 2t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q} - 2E\frac{\partial}{\partial E} - B\frac{\partial}{\partial B}$$

$$N_5 = -2t\frac{\partial}{\partial q} + 2q\frac{\partial}{\partial S} + 2B\frac{\partial}{\partial E} - 2\frac{\partial}{\partial B}$$

$$N_6 = 2t^2\frac{\partial}{\partial t} + 2qt\frac{\partial}{\partial q} + (\hbar t - q^2)\frac{\partial}{\partial S} - (2qB + 4tE + \hbar)\frac{\partial}{\partial E} + 2(q - tB)\frac{\partial}{\partial B}$$

$$N_g = e^{\frac{1}{\hbar}S}\left\{-\hbar g\frac{\partial}{\partial S} + (\hbar g_t - Eg)\frac{\partial}{\partial E} + (\hbar g_q - Bg)\frac{\partial}{\partial B}\right\}$$

$\forall\ g > 0$ solving (1)

For $\forall\ \delta,\delta'$ vector fields in M, def. Ω, bilinear, antisymmetric, with value in vector fields M :

(15) $$\Omega(\delta,\delta') = \left(\delta(B)\delta'(q) - \delta(q)\delta'(B)\right) + \left(\delta(E)\delta'(t) - \delta(t)\delta'(E)\right)$$

« Bilinear covariant » (Cartan)

$\forall$N in subspace of g generated by and $\forall$ vector field δ in M.

(16) $$\Omega(N,\delta) = -\delta\left(n_N\right)$$

where n_N is the function $N^t E + N^q B + N^S \equiv n_N$.

Def : Lagrangian (free) : $L(q,B) = \dfrac{B^2}{2}$.

Poincaré-Cartan form : $\omega_{pc} = \omega - dS = Bdq + Edt$

Section map (w.r.t. solution η of (1)) :

$$\theta_\eta(t) = t,\ \ \theta_\eta(q) = q,\ \ \ \theta_\eta(S) = -\hbar \ln \eta \equiv \tilde{S}(q,t)$$

$$\theta_\eta(E) = \hbar \frac{\partial_t \eta}{\eta} \equiv \tilde{E}(q,t),\ \ \ \theta_\eta(B) = \hbar \frac{\nabla \eta}{\eta} \equiv \tilde{B}(q,t).$$

Idea : Impose independence of (q,t), i.e. consider a 2d-submanifold of M +, where the forms $\in I_{HJB}$ are annulled.

Then

(17)

$$\begin{cases} \theta_\eta(\omega_{pc}) = \tilde{B}\, dq + \tilde{E}\, dt \\ \theta_\eta(n_N) = \hbar \dfrac{\nu_N \eta}{\eta} \quad \text{where} \quad \nu_N \equiv N^t \dfrac{\partial}{\partial t} + N^q \dfrac{\partial}{\partial q} + \dfrac{1}{\hbar} N^S \text{ (acting on (q,t))} \\ \theta_\eta \circ \Omega \equiv \Omega_\eta \end{cases}$$

Teorem : $\forall N, N' \in$ subalgebra (g) (Not transforming S variable)

(18) $\Omega_\eta(N,N') = \hbar \dfrac{\nu_{[N,N']}\eta}{\eta}$ and $d\Omega_\eta = 0.$

Probabilistic interpretation :

$$\tilde{S}(q,t) = E_{q,t} \int_t^T \omega_{pc} d\tau \text{ along solutions } \quad \begin{cases} dz_\tau = \hbar^{1/2} dw_\tau + \tilde{B}(z_\tau,\tau)\, d\tau \\ z_t = q \end{cases}$$

$$= E_{q,t} \int_t^T \frac{1}{2} \tilde{B}^2(z_\tau,\tau) d\tau.$$

(19) $\tilde{S}(q,t) \equiv E_{q,t} \int_t^T L_0(Dz_\tau)\, d\tau$

Non singular along $\tau \mapsto z_\tau$!

The geometrical variation of the Lagrangian :

(20) $\mathcal{L}_N(L_0) + L_0 \frac{dN^t}{dt} = -D_t N^S.$

$\rightarrow$ Calculus of variation for z_τ, almost without Itô calculus !

Example :

$$N = -\frac{1}{2} N_6$$

$$\hat{N} = -\frac{1}{2}\nu_{N_6} = -\frac{1}{2}\left(N_6^t \frac{\partial}{\partial t} + N_6^q \frac{\partial}{\partial q} + \frac{1}{\hbar} N_6^5\right) = t^2 \frac{\partial}{\partial t} + qt \frac{\partial}{\partial q} - \frac{1}{2\hbar}(q^2 - \hbar t)$$

$U_\alpha^N = e^{\alpha \hat{N}}$ maps

$$\eta(q,t) \mapsto \frac{1}{\sqrt{1-\alpha t}} \varepsilon^{-\frac{\alpha q^2}{2\hbar(1-\alpha t)}} \eta\left(\frac{q}{1-\alpha t}, \frac{t}{1-\alpha t}\right) \equiv \eta_\alpha(q,t)$$

$$(Q,T) \mapsto \left(\frac{Q}{1+\alpha T}, \frac{T}{1+\alpha T}\right)$$

If z_t built from $\eta(q,t)$,
z_t^α built from $\eta_\alpha(q,t)$

$$z^\alpha(t) = (1-\alpha t) z\left(\frac{t}{1-\alpha t}\right) \qquad \forall \alpha$$

Meaning when $z_t = \hbar^{1/2} w_t$ (i.e. $\eta_t = 1 \quad \forall t$) :

z_t^α solves SDE :

$$dz_t^\alpha = \hbar^{1/2} dw_t - \frac{\alpha z_t^\alpha}{1-\alpha t} dt$$

In some cases, z_t^α coincides with z_t, $\forall \alpha$.

For example, all invariances of Brownian, like

$w_t^\alpha = \alpha^{-1/2} w_{\alpha t}$, Projective invariance, etc…, follow.

Coming back to (free) quantum particle, our $\hat{N} \rightarrow$

$$qt \frac{\partial}{\partial q} + t^2 \frac{\partial}{\partial t} - \frac{1}{2}\left(q^2 + i\hbar t\right)$$

$$Q(t)t \circ P(t) - t^2 H_0(t) - \frac{1}{2}(Q^2(t) + i\hbar t) \equiv N(t)$$

where A(t) =observable in Heisenberg sense is a quantum cste of motion :

$$i\hbar \frac{\partial}{\partial t} N(t) + \left[N(t), H_0 \right] = 0$$

The approach provides many new informations on quantum systems and symmetries (but probabilistic content lost).

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