Stable Constant Mean Curvature Hypersurfaces Inside Convex Domains

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Abstract

Let $D$ be a convex domain in $\mathbb{R}^{n+1}$. In this work we first summarize known results about the problem of classifying compact, stable, constant mean curvature hypersurfaces inside $D$. Then, we add to the literature the solution to this problem when the domain is a solid convex cone $C$. As a consequence, we can solve the isoperimetric problem in $C$ consisting in finding a hypersurface separating a given amount of volume with the least possible area.

1. Statement of the problem

The notion of stability

Let $D \subset \mathbb{R}^{n+1}$ be a domain (connected, open set) with $\partial D \neq \emptyset$, and $\Sigma \subset D$ a smooth, compact hypersurface separating a bounded open set $\bar{E} \subset D$. Call $\operatorname{vol}(E)$ Lebesgue measure of $E$.

- A volume preserving variation (v.p.v.) of $\Sigma$ is a smooth family $\{\Sigma_t\}_{t \in (-\epsilon, \epsilon)} \subset D$ such that
  \( \dot{t} \Sigma_t = t \Sigma_t \) and $\operatorname{vol}(E)$ for any $t \in (-\epsilon, \epsilon)$, where $E_t$ is the open set bounded by $\Sigma_t$.
- We say that $\Sigma$ is stable if the area functional $A(t)$ associated to any v.p.v. of $\Sigma$ satisfies $A'(0) = 0$ and $A''(0) \geq 0$.

Questions related

- Obtain geometric and topological information about stable hypersurfaces inside $D$.
- Find those domains for which it is possible to classify all stable hypersurfaces.

2. Geometric and analytic description of stability

Theorem (Barbosa and do Carmo, Ros and Vergasta): $\Sigma \subset D$ is stable if and only if

- $\Sigma$ has constant mean curvature $H_0$.
- $\Sigma$ meets $\partial D$ orthogonally.
- $I(u)$ $\geq 0$ for any $u \in C^2(\Sigma)$ with $\int_{\Sigma} u dA = 0$, where $I$ is the index form of $\Sigma$.

- $- \nabla \cdot v$ is $\text{grad}$ relative to $\Sigma$.
- $\sigma_1^2 = \text{squared norm of the second fundamental form of } \Sigma$.
- $\Pi = \text{second fundamental form of } \partial D$ respect to the inner normal.
- $\nu = \text{unit normal along } \Sigma$ pointing into $E$.

Remark: The boundary term indicates us that stability is more restrictive if $D$ is convex.

Main idea: Insert an appropriate test function in the index form

3. Description of stable hypersurfaces in convex domains

This is an interesting and difficult problem. Here we show some known results for certain domains with geometric particularities. The complete solution for balls remains open!

- Half-spaces (Barbosa and do Carmo)
- Slabs in $\mathbb{R}^{n+2}$ ($n \leq 7$) (Pedrosa and Ritoré)
- Some stable $\Sigma$’s in balls (Ros and Vergasta)

Our contribution: Stable hypersurfaces in solid convex cones

Notation: Let $C = 0 \times U = \{0 \times t > 0, \eta \in U\}$ be the cone over the smooth domain $U \subset \mathbb{R}^n$.

Theorem (Ritoré, –): The only stable hypersurfaces in a convex cone $C \subset \mathbb{R}^{n+1}$ are

- round spheres contained in $C$,
- half-spheres in $C$ with boundary in a flat piece of $\partial C$,
- spherical caps centered at the vertex.

Important fact: The theorem above is true if we allow the presence in $\Sigma$ of a closed singular set which is negligible in the sense of Measure Theory (small Hausdorff codimension).

4. Applications to the isoperimetric problem

Definition: $\Sigma$ is an isoperimetric hypersurface in a domain $D$ if globally minimizes the area under the restriction of the volume it encloses.

- It is clear that any isoperimetric $\Sigma$ is stable.

How to use stability in the isoperimetric problem

1. Ensure existence of isoperimetric hypersurfaces in $D$.
2. This is a difficult problem if $D$ is unbounded.
3. Find candidates, classify stable hypersurfaces inside $D \rightarrow \Sigma$, candidates.
4. Compare the area of candidates to find the best ones.

Isoperimetric hypersurfaces in convex cones

Existence (Ritoré, –): If $C$ is a cone admitting a local support hyperplane at a point $y$ in $\partial C \setminus \{0\}$, then there are isoperimetric hypersurfaces enclosing any given amount of volume. Isoperimetric solutions may have a closed negligible singular set. We can apply our classification of stable hypersurfaces in a convex cone to find the stable candidates. An easy comparison among the areas of candidates finally gives us

Theorem (Lions and Pacella): Isoperimetric hypersurfaces in a solid convex cone are