Least-perimeter Partitions of the Disk

Antonio Cañete
Departamento de Geometría y Topología
Universidad de Granada

Murcia, September 20th, 2004
**Problem**

Partitioning a planar disk into $n$ regions of given areas with the least possible perimeter
Problem
Partitioning a planar disk into $n$ regions of given areas with the least possible perimeter

For $n = 2$ regions, the solution is given by

![Diagram of a planar disk partitioned into two regions](image)
Theorem ([CR], Th. 4.1)\(^1\)

Let \( C \) be a minimizing graph separating a disk \( D \) into three regions of given areas. Then \( C \) is a standard graph.

Existence of solution

Theorem ([M], Th. 2.3)²: There exists a least-perimeter graph separating $D$ into $n$ regions of prescribed areas. Moreover such a graph consists of

- smooth curves,
- meeting in threes in the interior of $D$ and
- meeting $\partial D$, one at a time

Such a graph is called *minimizing graph*

Notation

An *admissible graph* \( C \subset D \) consists of vertices and curves so that:

- At every interior vertex three curves meet
- At every vertex in \( \partial D \) just one curve meets \( \partial D \)
Notation

An *admissible graph* $C \subset D$ consists of vertices and curves so that:

- At every interior vertex three curves meet
- At every vertex in $\partial D$ just one curve meets $\partial D$

$C$ induces a decomposition of $D$ into $n$ regions $R_i$, possibly nonconnected
Notation

An **admissible graph** $C \subset D$ consists of vertices and curves so that:

- At every interior vertex three curves meet
- At every vertex in $\partial D$ just one curve meets $\partial D$

$C$ induces a decomposition of $D$ into $n$ regions $R_i$, possibly nonconnected

- $C_{ij} =$ curve separating regions $R_i$ and $R_j$
- $N_{ij} =$ normal vector to $C_{ij}$ pointing into $R_i$
- $h_{ij} =$ geodesic curvature of $C_{ij}$
Variational Formulae

For an admissible graph we will consider smooth variations
\[ \varphi_t : C \rightarrow D \]

Associated vector field:
\[ X = \left. \frac{d}{dt} \right|_{t=0} \varphi_t \]

The derivative of the area \( A_i \) of region \( R_i \) at \( t = 0 \) is given by
\[ \left. \frac{dA_i}{dt} \right|_{t=0} = - \sum_{j \in I(i)} \int_{C_{ij}} X \cdot N_{ij}, \]

where \( I(i) = \{ j \neq i; R_j \text{ touches } R_i \} \).
**First variation of length:** Consider an admissible graph $C \subset D$, and a smooth variation $\varphi_t : C \rightarrow D$ with associated vector field $X$. Then the first derivative of the length of $\varphi_t(C)$ at $t = 0$ is given by

$$\left. \frac{dL}{dt} \right|_{t=0} = -\frac{1}{2} \sum_{i \in \{1, \ldots, n\}} \int_{C_{ij}} h_{ij} u_{ij} + \sum_{p \in \partial C_{ij}} X(p) \cdot \nu_{ij}(p)$$

where:

- $u_{ij} = X \cdot N_{ij}$
- $\nu_{ij}(p)$ is the inner conormal to $C_{ij}$ in $p$
First variation of length: Consider an admissible graph $C \subset D$, and a smooth variation $\varphi_t : C \to D$ with associated vector field $X$. Then the first derivative of the length of $\varphi_t(C)$ at $t = 0$ is given by

$$
\left. \frac{dL}{dt} \right|_{t=0} = -\frac{1}{2} \sum_{i \in \{1, \ldots, n\}} \sum_{j \in I(i)} \left\{ \int_{C_{ij}} h_{ij} u_{ij} + \sum_{p \in \partial C_{ij}} X(p) \cdot \nu_{ij}(p) \right\},
$$

where:

- $u_{ij} = X \cdot N_{ij}$
- $\nu_{ij}(p)$ is the inner conormal to $C_{ij}$ in $p$

An admissible graph is **stationary** if $\left. \frac{dL}{dt} \right|_{t=0} = 0$ for any area-preserving variation.
First variation of length: Consider an admissible graph $C \subset D$, and a smooth variation $\varphi_t : C \to D$ with associated vector field $X$. Then the first derivative of the length of $\varphi_t(C)$ at $t = 0$ is given by

$$\left. \frac{dL}{dt} \right|_{t=0} = -\frac{1}{2} \sum_{\substack{i \in \{1,\ldots,n\} \setminus j \in I(i) \cap \partial C \cup \partial C_i}} \left\{ \int_{C_{ij}} h_{ij} u_{ij} + \sum_{p \in \partial C_{ij}} X(p) \cdot \nu_{ij}(p) \right\},$$

where:

- $u_{ij} = X \cdot N_{ij}$
- $\nu_{ij}(p)$ is the inner conormal to $C_{ij}$ in $p$

An admissible graph is **stationary** if $\left. \frac{dL}{dt} \right|_{t=0} = 0$ for any area-preserving variation

A minimizing graph must be stationary
Consequences:
Consequences:

- $h_{ij}$ is \textit{constant} on $C_{ij}$
Consequences:

- $h_{ij}$ is constant on $C_{ij}$
- The edges of $C$ meet in threes at 120-degree angles in interior vertices
Consequences:

- \( h_{ij} \) is **constant** on \( C_{ij} \)

- The edges of \( C \) meet in threes at **120-degree angles** in interior vertices

- **Balancing condition**: given three edges \( C_{ij}, C_{jk}, C_{ki} \) meeting in an interior vertex, their geodesic curvatures satisfy

  \[
  h_{ij} + h_{jk} + h_{ki} = 0
  \]
Consequences:

- $h_{ij}$ is \textit{constant} on $C_{ij}$
- The edges of $C$ meet in threes at \textit{120-degree angles} in interior vertices
- \textit{Balancing condition}: given three edges $C_{ij}, C_{jk}, C_{ki}$ meeting in an interior vertex, their geodesic curvatures satisfy
  \[ h_{ij} + h_{jk} + h_{ki} = 0 \]
- The edges of $C$ meet $\partial D$ \textit{orthogonally}
These *regularity conditions* give us the solution for $n = 2$:

![Least-perimeter partition of the disk into two regions of given areas](attachment:image.png)
Pressure of a region
Pressure of a region

The balancing condition

\[ h_{ij} + h_{jk} + h_{ki} = 0 \]

allows to define a pressure \( p_i \) on every region \( R_i \):

\[ p_i \in \mathbb{R} \text{ such that } h_{ij} = p_i - p_j \]
Second variation of length: Let $C$ be a stationary graph and let $\{\varphi_t\}$ be a variation preserving areas. Then the second derivative of length at $t = 0$ is given by

$$\left. \frac{d^2 L}{dt^2} \right|_{t=0} = -\frac{1}{2} \sum_{i=1,\ldots,n} \sum_{j \in I(i)} \int_{C_{ij}} (u''_{ij} + h_{ij}^2 u_{ij}) u_{ij}$$

$$+ \sum_{\substack{p \in \partial C_{ij} \cap \text{int}(D) \atop p \in \partial C_{ij} \cap \partial D}} \left( -q_{ij} u_{ij}^2 + u_{ij} \frac{\partial u_{ij}}{\partial \nu_{ij}} \right)(p) + \sum_{\substack{p \in \partial C_{ij} \cap \partial D \atop p \in \partial D}} \left( u_{ij}^2 + u_{ij} \frac{\partial u_{ij}}{\partial \nu_{ij}} \right)(p) \right\},$$

where

$$q_{ij}(p) = \frac{1}{\sqrt{3}} \left( h_{ki} + h_{kj} \right)(p)$$
For an area-preserving variation *by stationary graphs*,

\[
\frac{d^2L}{dt^2} = \sum_\alpha \frac{dp_\alpha}{dt} \frac{dA_\alpha}{dt},
\]

where \( \alpha \) labels the components of the regions.
For an area-preserving variation by stationary graphs,

$$\frac{d^2L}{dt^2} = \sum_{\alpha} \frac{dp_\alpha}{dt} \frac{dA_\alpha}{dt},$$

where $\alpha$ labels the components of the regions.

A stationary graph is **stable** if

$$\left. \frac{d^2L}{dt^2} \right|_{t=0} \geq 0$$

for any area-preserving variation.
For an area-preserving variation by stationary graphs,

\[ \frac{d^2L}{dt^2} = \sum_{\alpha} \frac{dp_{\alpha}}{dt} \frac{dA_{\alpha}}{dt}, \]

where \( \alpha \) labels the components of the regions

A stationary graph is \textit{stable} if

\[ \frac{d^2L}{dt^2} \bigg|_{t=0} \geq 0 \]

for any area-preserving variation

A minimizing graph must be stable
Bound on the number of components of $R_1$

Let $R_1$ be the region of largest pressure
Bound on the number of components of $R_1$

Let $R_1$ be the region of largest pressure

Lemma ([CR], Lem. 2.1): Let $C$ be a stable graph separating $D$ into $n$ regions. Then $R_1$ has at most $n - 1$ nonhexagonal components
Bound on the number of components of $R_1$

Let $R_1$ be the region of largest pressure

**Lemma ([CR], Lem. 2.1):** Let $C$ be a stable graph separating $D$ into $n$ regions. Then $R_1$ has at most $n - 1$ nonhexagonal components

**Consequence:** For $n = 3$ regions, we can find the non-standard possible minimizing graphs
The nine \textit{non-standard} possibilities for $n = 3$
The *last two* configurations can be ruled out by using a geometrical transformation, so they are not minimizing:

![Diagram](image-url)
Configurations *third* to *seventh* are unstable:

**Lemma ([HMRR], Prop. 5.2)**: Let \( C \subset D \) be a stationary graph. If there exists a Jacobi function with four nodal domains, then \( C \) is unstable

We apply the above result considering the rotations vector field

\[\begin{array}{c}
\beta \\
\alpha \\
\beta \\
3 \\
\end{array}\]
Configurations *third* to *seventh* are unstable:

Lemma ([HMRR], Prop. 5.2)\(^3\): Let \( C \subset D \) be a stationary graph. If there exists a Jacobi function with four nodal domains, then \( C \) is unstable.

We apply the above result considering the rotations vector field.

---

\[
\begin{array}{c}
\beta \\
1 \\
\alpha \\
1 \\
\alpha \\
\end{array}
\]
The *first two* configurations are also unstable:

Both have a region with *two triangles touching* $\partial D$
Proposition ([CR], Prop. 3.4): Given a stationary graph $C$ with a triangle $\Omega$ touching $\partial D$, there exists a variation by stationary graphs of $C$ that:
Proposition ([CR], Prop. 3.4): Given a stationary graph $C$ with a triangle $\Omega$ touching $\partial D$, there exists a variation by stationary graphs of $C$ that:

- increases the area of $\Omega$
Proposition ([CR], Prop. 3.4): Given a stationary graph $C$ with a triangle $\Omega$ touching $\partial D$, there exists a variation by stationary graphs of $C$ that:

- increases the area of $\Omega$
- decreases the pressure of $\Omega$, keeping the other pressures unchanged
Proposition ([CR], Prop. 3.4): Given a stationary graph $C$ with a triangle $\Omega$ touching $\partial D$, there exists a variation by stationary graphs of $C$ that:

- increases the area of $\Omega$
- decreases the pressure of $\Omega$, keeping the other pressures unchanged
- leaves invariant the edges of $C$ not lying in $\Omega$
Proposition ([CR], Prop. 3.4): Given a stationary graph $C$ with a triangle $\Omega$ touching $\partial D$, there exists a variation by stationary graphs of $C$ that:

- increases the area of $\Omega$
- decreases the pressure of $\Omega$, keeping the other pressures unchanged
- leaves invariant the edges of $C$ not lying in $\Omega$

For this variation, in $\Omega$ we have

$$\frac{dp}{dt} \cdot \frac{dA}{dt} < 0$$
Proposition ([CR], Prop. 3.5): Let $C$ be a stationary graph in which a region has two triangles $\Omega_1, \Omega_2$ touching $\partial D$. Then $C$ is unstable.
Proposition ([CR], Prop. 3.5): Let $C$ be a stationary graph in which a region has two triangles $\Omega_1, \Omega_2$ touching $\partial D$. Then $C$ is unstable.

Proof:

i) In each triangle, we consider the variation of the above Proposition.
Proposition ([CR], Prop. 3.5): Let $C$ be a stationary graph in which a region has two triangles $\Omega_1, \Omega_2$ touching $\partial D$. Then $C$ is unstable.

Proof:

i) In each triangle, we consider the variation of the above Proposition.

ii) Combining both of them, we can obtain another variation by stationary graphs, preserving the areas.
Proposition ([CR], Prop. 3.5): Let $C$ be a stationary graph in which a region has two triangles $\Omega_1, \Omega_2$ touching $\partial D$. Then $C$ is unstable.

Proof:

i) In each triangle, we consider the variation of the above Proposition.

ii) Combining both of them, we can obtain another variation by stationary graphs, preserving the areas.

iii) For this new variation,

$$
\left. \frac{d^2 L}{dt^2} \right|_{t=0} = \sum_{\alpha} \frac{dp_\alpha}{dt} \frac{dA_\alpha}{dt}
$$
Proposition ([CR], Prop. 3.5): Let $C$ be a stationary graph in which a region has two triangles $\Omega_1, \Omega_2$ touching $\partial D$. Then $C$ is unstable.

Proof:

i) In each triangle, we consider the variation of the above Proposition.

ii) Combining both of them, we can obtain another variation by stationary graphs, preserving the areas.

iii) For this new variation,

$$\left.\frac{d^2 L}{dt^2}\right|_{t=0} = \sum_{\alpha} \frac{dp_\alpha}{dt} \frac{dA_\alpha}{dt} = \frac{dp_1}{dt} \frac{dA_1}{dt} + \frac{dp_2}{dt} \frac{dA_2}{dt} < 0,$$

so the graph is unstable.
Then the only remaining configuration is the *standard graph*.

For given areas, there is *uniqueness* up to rigid motions.
Open questions
Open questions

- **Conjecture 1:** A minimizing graph separates the disk into connected regions
Open questions

• Conjecture 1: A minimizing graph separates the disk into connected regions

• Conjecture 2: For \( n = 4 \) regions, the minimizing graph is given by
Conjecture 3: For $n = 5$ regions, the minimizing graph is given by
• Conjecture 4: For $n = 6$ regions, the minimizing graph is given by