Bezout Inequality for Mixed volumes.

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(based on joint works with Christos Saroglou and Ivan Soprunov)

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- We will often use notion of Minkowski sum: $K + L = \{x + y : x \in K \text{ and } y \in L\}$.
- We all know that $V_n(\lambda K) = \lambda^n V_n(K)$ for $\lambda \geq 0$, i.e. volume is a homogeneous measure of degree of homogeneity n. But there is much more!!!

Main Definitions: Mixed Volume

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- If $K \subset L$, then $V(K, K_2, K_3, \dots, K_n) \leq V(L, K_2, K_3, \dots, K_n)$.

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so if $V_n(K) = V_n(rB_2^n)$, for some $r > 0$, then $V_{n-1}(\partial K) \geq V_{n-1}(r\mathbb{S}^{n-1})$.

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- Alexandrov–Fenchel inequality:

$$V(K_1, K_2, K_3, \ldots, K_n) \geq \sqrt{V(K_1, K_1, K_3, \ldots, K_n) V(K_2, K_2, K_3, \ldots, K_n)}.$$

Main Topic (for those who do not care about introduction)

Question 1

Fix an integer $2 \le r \le n$ and let $D \subset \mathbb{R}^n$ be a convex body which satisfies

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for all convex bodies $K_1, \dots, K_r \subset \mathbb{R}^n$. Is it true that then D must be n-simplex?

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Plan

- How one could come up with such inequalities & why they are (may be) interesting?

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Question 2

What is the best constant $c_{n,r}$ such that

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is true for all convex bodies $K_1, \ldots, K_r$ and D in $\mathbb{R}^n$?

Plan

- How one could come up with such inequalities & why they are (may be) interesting?
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Main Topic (for those who do not care about introduction)

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Let $X_1, \dots X_n \subset \mathbb{C}^n$ be hypersurfaces defined by polynomials $F_1, \dots, F_n$:

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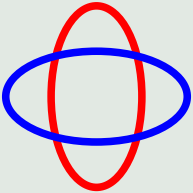
Childish Example: Two quadratic polynomials.

$$F_1(x,y) = \frac{x^2}{9} + \frac{y^2}{60} - 1 \quad \text{and} \quad F_2 = \frac{x^2}{50} + \frac{y^2}{2} - 2.$$



Then $\deg F_1 = \deg F_2 = 2$ and X_1, X_2 are ellipses which intersect in exactly 4 points.

Newton Polytope

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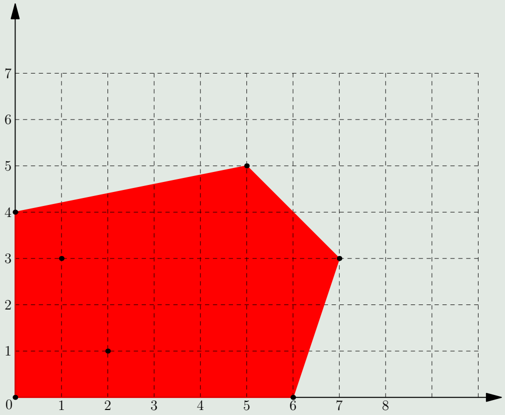
$F(x,y) = 4x^7y^3 - 5x^5y^5 + 13x^6 - 5y^4 + 21x^2y + 13xy^3 - 71$



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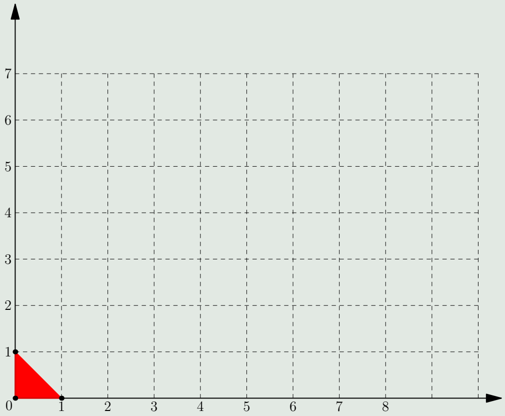
Interesting case - affine function $F(x,y) = 3x - 15y + 71$



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Let $F_1, \ldots, F_n$ be polynomials with fixed Newton Polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$ and generic coefficients. Then

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$$\deg(F_i) = n! V(P_i, \Delta[n-1]).$$

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Fix integer $2 \leq r \leq n$ and let Δ any n-dimensional simplex, then

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- There are indecomposable bodies for which the inequality is not true: $D = B_1^3$.

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- (C. Saroglou, I. Soprunov & A.Z., 2016): If D is a **polytope** then $D = \Delta$.

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Let D be an n-dimensional convex body which satisfies

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$$W(g) = \bigcap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \le g(u)\}.$$

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where $D|\xi^\perp$ denotes the orthogonal projection of D onto the hyperplane orthogonal to ξ. In addition, assume $\xi\cdot\nu=0$. Then, similarly, for the orthogonal projection we can compute the volume of $D|(\xi,\nu)^\perp$:

$$V_{n-2}(D|(\xi,\nu)^\perp)=n(n-1)V(K_1,K_2,D[n-2]).$$

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$$V_{n-2}(D|(\xi, \nu)^\perp) = n(n-1) V(K_1, K_2, D[n-2]).$$

Substituting the above calculations in inequality in Question 1, we get

$$\frac{n}{n-1} V_{n-2}(D|(\xi, \nu)^\perp) V_n(D) \leq V_{n-1}(D|\xi^\perp) V_{n-1}(D|\nu^\perp).$$

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Moving towards Question 2 & connections to projections

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Assume Z_1, Z_2 are zonoids, then

$$V(Z_1, Z_2, D[n-2]) V_n(D) \leq \mathbf{2} V(Z_1, D[n-1]) \cdot V(Z_2, D[n-1])$$

for any convex, symmetric body D.

Question 2: the case of zonoids.

I. Soprunov & A.Z.; 2016

Suppose D is a convex body in $\mathbb{R}^n$ and $Z_1, \dots Z_r$ are zonoids then

$$V(Z_1, \dots, Z_r, D^{n-r}) V_n(D)^{r-1} \leq \frac{r^r}{r!} \prod_{i=1}^{r} V(Z_i, D^{n-1}),$$

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Idea of the proof: Use ideas of Giannopoulos, Hartzoulaki; 2002 & Paouris / Fradelizi, Giannopoulos & Meyer; 2003: apply the Berwald's Lemma to prove that if $D \subset \mathbb{R}^n$ is a convex body, then

$$\left(\frac{n}{r}\right)^r \binom{n}{r}^{-1} V_{n-r}(D|(e_1, e_2, \dots, e_r)^{\perp}) V_n(D)^{r-1} \leq \prod_{i=1}^{r} V_{n-1}(D|e_i^{\perp}).$$

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Next use multi-linearity and other properties of mixed volume to bring it back to zonoids.

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There exists a constant $c_{n,r} \leq n^r r^r / r!$ such that

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holds for all convex bodies $K_1, \dots, K_r$ and D in $\mathbb{R}^n$. Moreover $c_{n,r} \leq n^{r/2} r^r / r!$ when $K_1, \dots, K_r$ are symmetric with respect to the origin.

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Reminder: We proved before that for symmetric, convex sets $K_1, K_2 \subset \mathbb{R}^2$ (note - K_1, K_2 are zonoids) we have

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There were a number of works on this inequality after ...**and before** our work!
Reminder: We proved before that for symmetric, convex sets $K_1, K_2 \subset \mathbb{R}^2$ (note - K_1, K_2 are zonoids) we have

$$V(K_1, K_2) V_2(D) \leq \mathbf{2} V(K_1, D) \cdot V(K_2, D).$$

I. Soprunov, A.Z.; 2016 / S. Artstein-Avidan, D. Florentin & Y. Ostrover; 2014 / M. Fradelizi, A. Giannopoulos & M. Meyer, (2003)

Assume K_1, K_2, D are convex bodies in $\mathbb{R}^2$ (i.e. **Not necessary symmetric!**) then

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Question 2: General Case.

I. Soprunov & A.Z.; 2016

There exists a constant $c_{n,r} \le n^r r^r / r!$ such that

$$V(K_1, \dots, K_r, D^{n-r}) V_n(D)^{r-1} \le c_{n,r} \prod_{i=1}^{r} V(K_i, D^{n-1})$$

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