On Brunn-Minkowski inequalities in product Metric Measure Spaces

J. Yepes Nicolás

Universidad de Murcia

(joint work with M. Ritoré)

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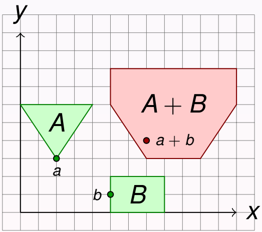
Here $\mathrm{vol}(\cdot)$ is the Lebesgue Measure and $A + B = \{a + b : a \in A, b \in B\}$ denotes the Minkowski sum of A and B.



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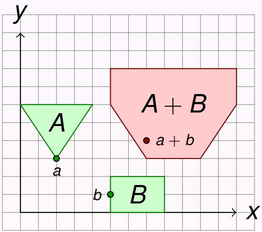
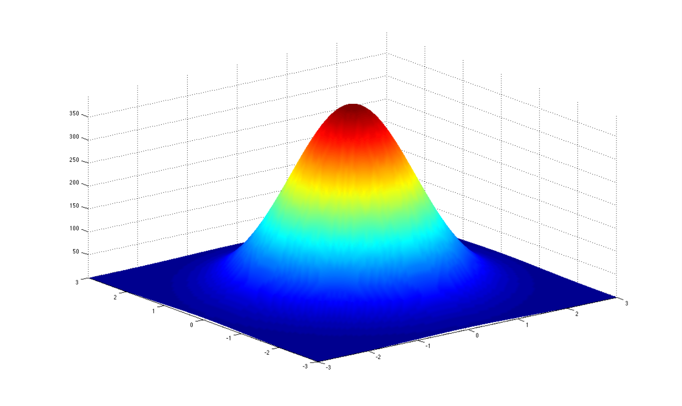


$\rightarrow$ It yields the isoperimetric inequality in a few lines: *Among all sets with a fixed surface area measure, Euclidean balls maximize the volume.*

The Gaussian Brunn-Minkowski inequality

The standard Gaussian measure in $\mathbb{R}^n$

$$\mathrm{d}\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} e^{\frac{-|x|^2}{2}} \mathrm{d}x, \quad x \in \mathbb{R}^n.$$



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2. $(1/n)$-concave version:

$$\gamma_n((1-\lambda)K + \lambda L)^{1/n} \geq (1-\lambda)\gamma_n(K)^{1/n} + \lambda\gamma_n(L)^{1/n}?$$

Not true (in general)! $\rightsquigarrow$ Special classes of sets must be considered.

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Conjecture (Gardner, Zvavitch (2010))

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Theorem (Gardner, Zvavitch (2010))

It is true when:

- the dimension $n = 1$.
- A and B are coordinate boxes containing the origin.
- Either A or B is a slab containing the origin.

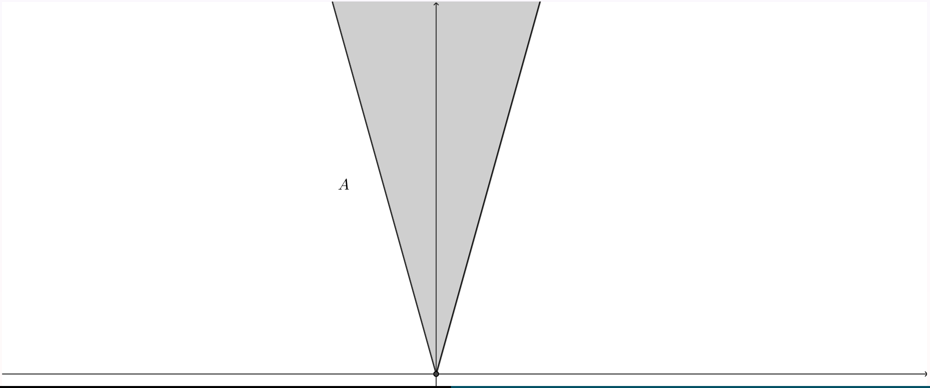
The Gaussian Brunn-Minkowski inequality

Counterexample (Nayar, Tkocz (2013))

The above conjecture is in general not true: it is enough to consider

$$A = \{(x,y) \in \mathbb{R}^2 : \ y \geq |x| \tan\alpha\}, \quad B = A + (0,-\varepsilon),$$

for $\varepsilon > 0$ small enough and $\alpha < \pi/2$ sufficiently close to $\pi/2$.



B-M ineqs for product of measures with certain concavity

Theorem (Livshyts, Marsigliettі, Nayar, Zvavitch (2017))

Let $\mu = \mu_1 \times \cdots \times \mu_n$ be a product measure on $\mathbb{R}^n$ such that μ_i is the measure given by $\mathrm{d}\mu_i(x) = \phi_i(x)\,\mathrm{d}x$, where $\phi_i : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ is a positively decreasing even function, $i = 1, \ldots, n$.
Let $\lambda \in (0, 1)$ and let $\emptyset \neq A, B \subset \mathbb{R}^n$ be unconditional measurable sets so that $(1-\lambda)A + \lambda B$ is also measurable. Then

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Unconditional sets

$A \subset \mathbb{R}^n$ is unconditional if for every $(x_1, \dots, x_n) \in A$ and every $(\epsilon_1, \dots, \epsilon_n) \in [-1,1]^n$ one has

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$\rightarrow$ In the same paper the authors pose the following question: *'can one remove the assumption of unconditionality in the Gaussian B-M inequality?'*

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Fact (the 1-dimensional case)

Let μ be the measure on $\mathbb{R}$ given by $\mathrm{d}\mu(x) = \phi(x)\mathrm{d}x$, where $\phi : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ is positively decreasing. Let $\lambda \in (0, 1)$ and let $A, B \subset \mathbb{R}$ be measurable sets with $0 \in A \cap B$ and such that $(1 - \lambda)A + \lambda B$ is also measurable. Then

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$\rightarrow$ Can one exploit this one-dimensional inequality to get a 'positive' answer to the above-mentioned question?

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We notice that, given $x, y, z \in \mathbb{R}^n$ and $\lambda \in (0,1)$, the relation $z = (1-\lambda)x + \lambda y$ holds if and only if

$$|z - x| = \lambda |x - y|, \; |z - y| = (1-\lambda) |x - y|.$$

Thus, we may extend

$$(1-\lambda)A+\lambda B=\{(1-\lambda)a+\lambda b:\ a\in A,\ b\in B\}$$

to the context of a metric space in the following way:

Definition

Let (X,d) be a metric space. If $\lambda\in(0,1)$ and given A, B two nonempty subsets of X, the "d-convex combination" $(1-\lambda)A\star_d\lambda B$ of A and B will be the nonempty set given by

$$(1-\lambda)A\star_d\lambda B=\bigl\{z\in X\colon d(z,a)=\lambda d(a,b),\ d(z,b)=(1-\lambda)d(a,b),\ a\in A, b\in B\bigr\}.$$

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From now on (X,d) will denote a metric space where d is a strictly intrinsic distance, i.e., such that for any $x,y\in X$ the closed balls $\overline{B}_d(x,r_1)$, $\overline{B}_d(y,r_2)$ have a nonempty intersection provided that $r_1+r_2=d(x,y)$.

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Main idea: exploiting some B-M inequalities in "simple spaces" in order to obtain new further B-M inequalities on "more involved" ones.

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- we will say that it *satisfies the Brunn-Minkowski inequality* with respect to $p \in \mathbb{R} \cup \{\pm\infty\}$ ("BM(p)" for short) if

$$\mu(C) \geq \left((1-\lambda)\mu(A)^p + \lambda\mu(B)^p\right)^{1/p}$$

  holds for all $\lambda \in (0,1)$ and any measurable sets A, B, C with $\mu(A)\mu(B) > 0$ such that $C \supset (1-\lambda)A \star_d \lambda B$.

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- If, further, in the above definition one may consider measurable sets A, B with $\mu(A)\mu(B) = 0$, we will say that (X, d, μ) *satisfies the general Brunn-Minkowski inequality* with respect to $p \in \mathbb{R} \cup \{\pm\infty\}$ ("$\overline{\mathrm{BM}}(p)$" for short).

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- In the same way, we will say that a certain family $\mathcal{F} \subset \mathcal{P}(X)$ satisfies $\mathrm{BM}(p)$ (resp. $\overline{\mathrm{BM}}(p)$) if the above definition is true when dealing with measurable sets $A, B, C \in \mathcal{F}$.

The product metric

One can define a distance $d_{X\times Y}$ on $X\times Y$ (whose induced topology agrees with the product topology) as follows:

$$\rho\big((x_1,x_2),(y_1,y_2)\big)=||(d_X(x_1,y_1),d_Y(x_2,y_2))||,$$

where $||\cdot||$ is a norm in $\mathbb{R}^2$.

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Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let ρ be the product metric on $X \times Y$. If A, B are non-empty subsets of $X \times Y$ and $\lambda \in (0,1)$ then $(1-\lambda)A \star_{d_{X\times Y}} \lambda B \subset (1-\lambda)A \star_\rho \lambda B$.

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With all this notation, our main result reads as follows:

Theorem (Ritoré, Y. N. (2018))

Let (X, d_X, μ_X), (Y, d_Y, μ_Y) be metric measure spaces where μ_X is σ-finite, μ_Y is locally finite and σ-finite, and $\mu_{X\times Y}$ is Radon.
If (X, d_X, μ_X), (Y, d_Y, μ_Y) satisfy $\overline{\mathrm{BM}}(1)$ and $\mathrm{BM}(p)$, respectively, for some $p \geq -1$, then $(X \times Y, d_{X\times Y}, \mu_{X\times Y})$ satisfies $\mathrm{BM}\big(1/(1/p+1)\big)$.

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Key message: the linear Brunn-Minkowski inequality, $\overline{\mathrm{BM}}(1)$, allows us to obtain further inequalities in other spaces. Moreover, it plays a relevant role along our approach in the sense that it cannot be replaced (in general) by another Brunn-Minkowski type inequality.

Corollary (Ritoré, Y. N. (2018))

Let (X, d_X, μ_X), (Y, d_Y, μ_Y) be metric measure spaces for which there exist certain families $\mathcal{F}_X \subset \mathcal{P}(X)$, $\mathcal{F}_Y \subset \mathcal{P}(Y)$ that satisfy $\overline{\mathrm{BM}}(1)$ and $\mathrm{BM}(p)$, respectively, where $p \geq -1$ and μ_X, μ_Y are σ-finite. Let $A, B \subset X \times Y$ be measurable sets such that $(1-\lambda)A \star \lambda B$ is so for $\lambda \in (0, 1)$. If moreover A, B satisfy:

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1. $0 < |\mu_Y(A(\cdot))|_\infty, |\mu_Y(B(\cdot))|_\infty < +\infty$,

B-M inequalities in Product Metric Measure Spaces

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Let (X, d_X, μ_X), (Y, d_Y, μ_Y) be metric measure spaces for which there exist certain families $\mathcal{F}_X \subset \mathcal{P}(X)$, $\mathcal{F}_Y \subset \mathcal{P}(Y)$ that satisfy $\overline{\mathrm{BM}}(1)$ and $\mathrm{BM}(p)$, respectively, where $p \geq -1$ and μ_X, μ_Y are σ-finite. Let $A, B \subset X \times Y$ be measurable sets such that $(1-\lambda)A \star \lambda B$ is so for $\lambda \in (0,1)$. If moreover A, B satisfy:

1. $0 < |\mu_Y(A(\cdot))|_\infty, |\mu_Y(B(\cdot))|_\infty < +\infty$,
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then

$$\mu_{X\times Y}\big((1-\lambda)A \star \lambda B\big) \geq M_{1/(1+p^{-1})}\big(\mu_{X\times Y}(A), \mu_{X\times Y}(B), \lambda\big).$$

B-M ineqs for product of measures with certain concavity

Definition

$A \subset \mathbb{R}^n$ is weakly unconditional if for every $(x_1, \ldots, x_n) \in A$ and every $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$ one has

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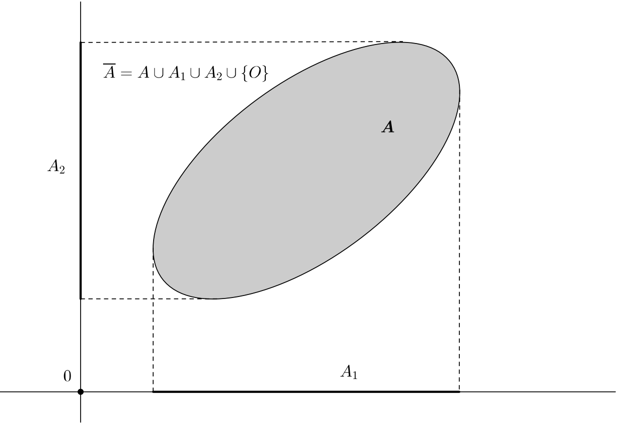

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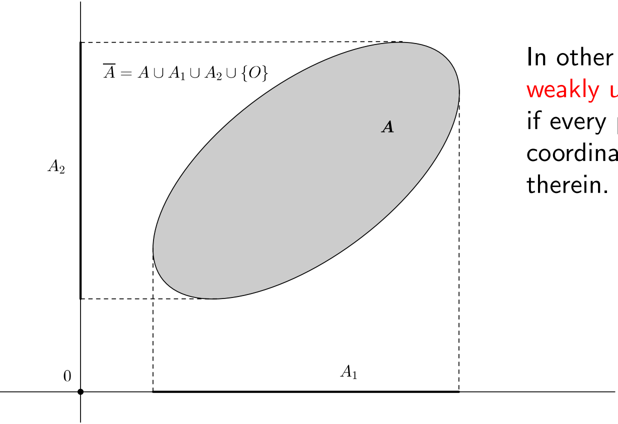

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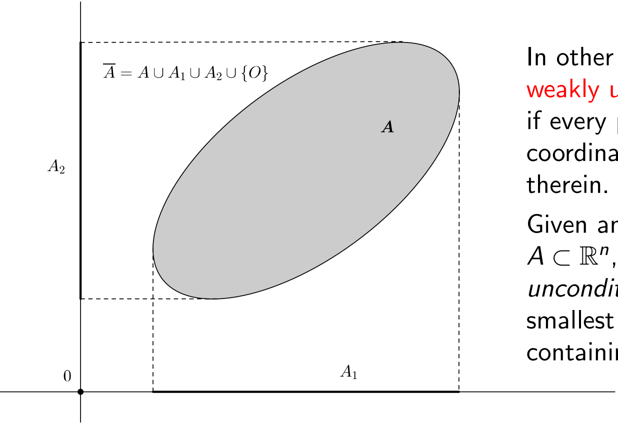

In other words, a set A will be weakly unconditional if and only if every projection onto any coordinate plane is contained therein.

Given an arbitrary nonempty set $A \subset \mathbb{R}^n$, $\overline{A}$ will denote its *weakly unconditional hull*, i.e., the smallest weakly unconditional set containing A.

B-M ineqs for product of measures with certain concavity

Theorem (Ritoré, Y. N. (2018))

Let $\mu = \mu_1 \times \cdots \times \mu_n$ be a product measure on $\mathbb{R}^n$ such that μ_i is the measure given by $\mathrm{d}\mu_i(x) = \phi_i(x)\,\mathrm{d}x$, where $\phi_i : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ is a positively decreasing function, $i = 1, \dots, n$.
Let $\lambda \in (0,1)$ and let $\emptyset \neq A, B \subset \mathbb{R}^n$ be weakly unconditional measurable sets such that $(1-\lambda)A + \lambda B$ is also measurable. Then

$$\mu((1-\lambda)A + \lambda B)^{1/n} \geq (1-\lambda)\mu(A)^{1/n} + \lambda\mu(B)^{1/n}.$$

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Corollary (Gaussian B-M inequality)

Let $\lambda \in (0,1)$ and let $\emptyset \neq A, B \subset \mathbb{R}^n$ be measurable sets such that $(1-\lambda)\overline{A} + \lambda\overline{B}$ is also measurable. Then

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$\rightarrow$ Notice that the sets in the counterexample by Nayar and Tkocz contain the origin and furthermore their projection onto the y-axis, and thus the sole "missing points" are those belonging to the x-axis.

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$\rightarrow$ Notice that weakly unconditional sets are those which *contain the origin in each of the n-linearly independent directions given by the coordinate axes*.

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3. $(\mathbb{R}_{>0}, d)$ satisfies the Prékopa-Leindler inequality, i.e., for any non-negative measurable functions $f, g, h : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{\geq 0}$ such that

   $$h((1-\lambda)x +_d \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda},$$

   then

   $$\int_{\mathbb{R}_{>0}} h \,\mathrm{d}x \geq \left(\int_{\mathbb{R}_{>0}} f \,\mathrm{d}x\right)^{1-\lambda} \left(\int_{\mathbb{R}_{>0}} g \,\mathrm{d}x\right)^{\lambda}.$$

Another application

Since $A^{1-\lambda}B^{\lambda} = (1-\lambda)A \star_{d_{X^n}} \lambda B$, for $A, B \subset \mathbb{R}^n_{>0}$ (as a product space of $(\mathbb{R}, d)$), we get

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Let $\lambda \in (0,1)$ and let $A, B \subset \mathbb{R}^n_{>0}$ be measurable sets such that $A^{1-\lambda}B^{\lambda}$ is also measurable. Then

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As a consequence, the log-Brunn-Minkowski inequality holds for unconditional convex bodies (Saroglou 2015):

$$\mathrm{vol}\big((1-\lambda)K +_0 \lambda L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}.$$

On Brunn-Minkowski inequalities in product Metric Measure Spaces

J. Yepes Nicolás

Universidad de Murcia

(joint work with M. Ritoré)

Summer School 2018:

New perspectives on Convex Geometry

Castro Urdiales

September 3rd, 2018