Angular Curvature Measures

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Federer's curvature measures

$P \subset \mathbb{R}^n$ convex polytope, $U \subset \mathbb{R}^n$ Borel set

$$\Phi_i(P,U) = \sum_F \gamma(F,P)\,\mathrm{vol}_i(F \cap U), \qquad i = 0,\ldots,n$$

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$$\gamma(F,P) = \mathrm{vol}_{n-k-1}(N_F P \cap S^{n-1})/\omega_{n-k-1}$$

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Every such curvature measure is called **angular**.

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Observation. For every f

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Alesker, Abardia, Bernig, Böröczky, Colesanti, Faifman, Haberl, Li, Ma, Parapatits, Saorin Gomez, Schuster, Wannerer,...

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- $0 \leq i < n-1$: constant coefficient curvature measures are angular (Bernig-Fu-Solanes '15)

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- Hermitian integral geometry provides further geometrically interesting examples

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- Even functions on $\widetilde{\mathrm{Gr}}_i(\mathbb{R}^n)$ correspond bijectively to functions on $\mathrm{Gr}_i(\mathbb{R}^n)$ Grassmannian of i-dim linear subspaces
- the **Plücker embedding** $E \mapsto e_1 \wedge \cdots \wedge e_i$ embeds $\widetilde{\mathrm{Gr}}_i(\mathbb{R}^n)$ into $\bigwedge^i \mathbb{R}^n$.

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and coincides with the space of constant coefficient curvature measures.

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Proof of Theorem A. If

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defines an angular curvature measure, then f satisfies the hypothesis of the Proposition. □

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$\mathcal{V}(M)$, $\mathcal{C}(M)$ smooth valuations and curvature measures

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- $\mathcal{V}(M)$ acts on $\mathcal{C}(M)$ in a natural way compatible with the product of valuations
- if $f\colon M \to M'$ is a smooth immersion and $\mu, \nu \in \mathcal{V}(M'), \Psi \in \mathcal{C}(M')$ then

$$(f^*\mu) \cdot (f^*\nu) = f^*(\mu \cdot \nu), \quad (f^*\mu) \cdot (f^*\Psi) = f^*(\mu \cdot \Psi).$$

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Theorem C. If $f\colon M \to M'$ is an isometric immersion of Riem. mfds, then

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Alesker product

Theorem (Fu '16)**.** Suppose $X \subset M$ is a compact submanifold with corners, and $T \times M \to M$ is a smooth proper family of diffeomorphisms $\varphi_t : M \to M,\ t \in T$, equipped with a smooth measure dt. Suppose further that the map $T \times S^*M \to S^*M$, induced by the derivative maps $\varphi_{t*} : S^*M \to S^*M$, is a submersion. Then

$$\mu(P) = \int_T \chi(\varphi_t(X) \cap P)\, dt$$

defines a smooth valuation on M. Given $\nu \in \mathcal{V}(M), \Psi \in \mathcal{C}(M)$ we have

$$(\mu \cdot \nu)(P) = \int_T \nu(\varphi_t(X) \cap P)\, dt,$$
$$(\mu \cdot \Psi)(P, E) = \int_T \Psi(\varphi_t(X) \cap P, E)\, dt.$$