Reciprocity and Indicatrices in Convexity

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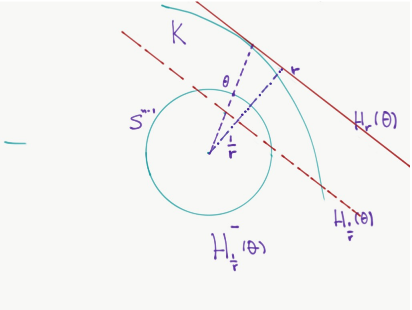
Spain, September 2018

$1/K$ vs. K°

We often discussed that the polar body K° may/should be visualized ("identified") with $1/K$. Let us directly construct this function "$1/K$". We fixed a scalar product and $0 \in K$. Then

$$K^\circ = \left\{ x \in \mathbb{R}^n \mid (x, y) \leq 1 \text{ for } \forall\, y \in K \right\}.$$



For hyperplane $H_r(\theta) = \{(x, \theta) = r\}$, $\theta \in S^{n-1}$, which supports the body K and has the distance r from 0, take another parallel hyperplane on the distance $1/r$ from 0. Consider the half-space $H^-_{1/r}(\theta)$ containing 0.

Then

$$K' = \bigcap_{\theta} H^{-}_{1/r}(\theta).$$

Let us call K' a reciprocal body of K.

Then $K' \subseteq K^{\circ}$ and $K'' \supseteq K$. Also $K''' = K'$, i.e.

Fact 1. K' is the duality on the image of this operation, i.e. on the family of reciprocal bodies.

So, $\exists$ a different "duality" on $\mathbb{R}^n$ but which is not surjective.

Note:

$$K = K' \iff K = B_2^n.$$

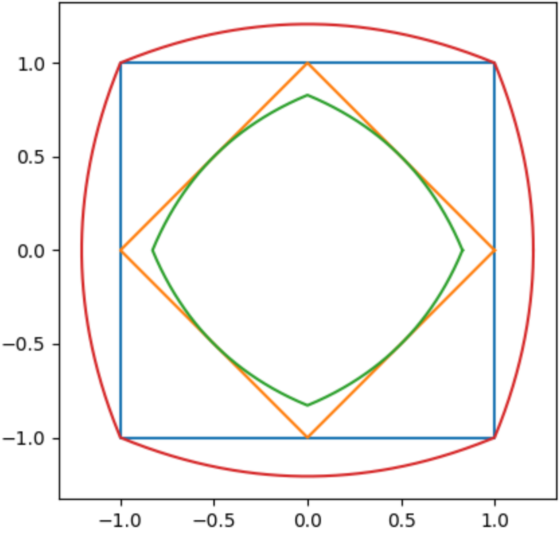
(For us K' is not a familiar body for most K.)

Note:

(i) $K' = K^{\circ} \Leftrightarrow B_2^n$;
(ii) $K^{\circ\prime} \subset K \subset K'^{\circ}$;

and equality holds from either side implies $K = B_2^n$.

In the next pictures: the started "chain" body K is in blue, K° – orange, K' – green and K'' – red.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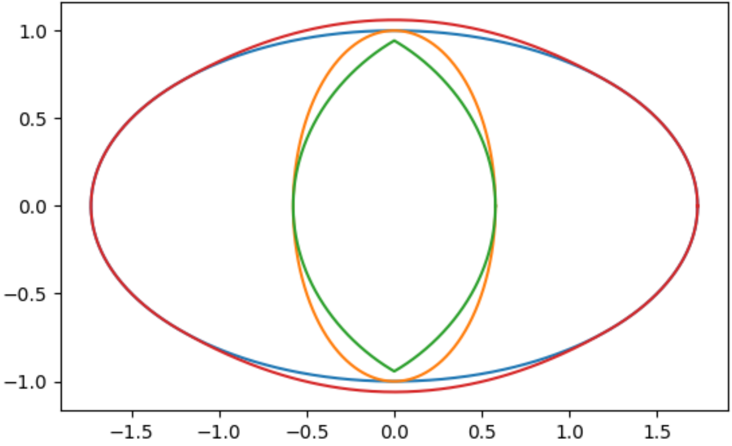
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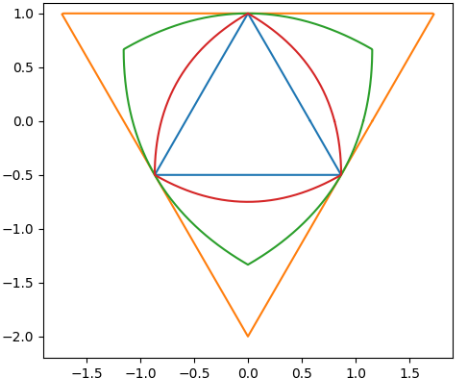
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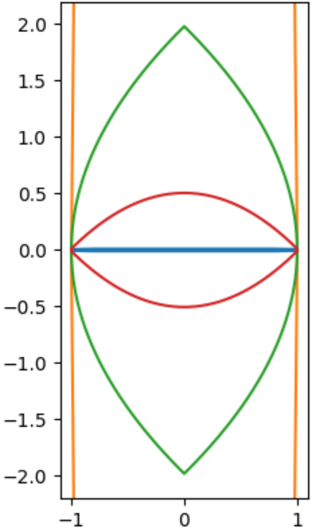
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Let us introduce an alternative description for K'. Introduce first the indicatrix K^* of the family of supporting functionals $\{h_K(\theta)\}$, $\theta \in S^{n-1}$:

The radial function $r_{K^*}(\theta) = h_K(\theta)$. K^* is the star body, $K^* \supseteq K$ and $K^* = K \Leftrightarrow K = B_2^n$.

Fact 2. $\forall\, K$, $(K^*)^\circ = K'$!

Let us introduce our notation (in the style of operators):

$$\circ : K \to K^\circ \text{ (i.e. } \circ K \equiv K^\circ)$$
$$* : K \to K^* \text{ (i.e. } * K \equiv K^*)$$
$$\prime : K \to K' \text{ (i.e. } \prime K \equiv K')$$

Theorem 1

Let $A' = B' (= K)$ *for convex A and B. Then* $(\lambda A + (1-\lambda)B)' = K, \forall \lambda : 0 \leq \lambda \leq 1$.

Let $K' = T$. Define by $D(T)$ the family of all convex bodies $\{A\}$ s.t. $A' = T$.

Theorem 1a

1. $\forall T$, $D(T)$ *is a closed convex subset of* $\mathcal{K}_0$.

2. $\forall T$, *s.t.* $D(T) \neq \emptyset$, T' *is the maximal in* $D(T)$, *if* K *s.t.* $K' = T$ *then* $K \subset T'$.

(See picture of the cube.)

Fact 3. $\forall K$, convex, $0 \in K$,

$$K^* = \cup\{B_x \mid x \in \partial K\} \equiv \cup\{B_x \mid x \in K\}$$

where B_x is the euclidean ball centered at $x/2$ and radius $|x|/2$ (i.e. $B(x/2, |x|/2)$), i.e. the interval $[0, x)$ is the diameter of B_x.

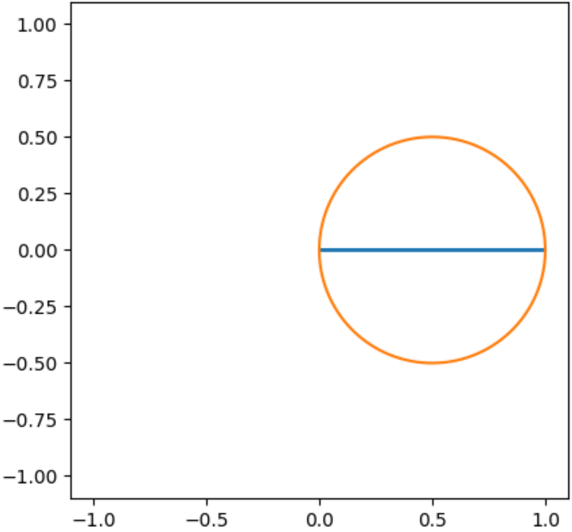
So each B_x passes through 0.

For interval $I = [0, x]$, $I^* = B_x$.

(In the next pictures the body K is blue and K^* is orange.)

We also extend the operation $*$ to the class of star-bodies. For A-star-body

$$*A := A^* = \cup\{B_x \mid x \in A\}.$$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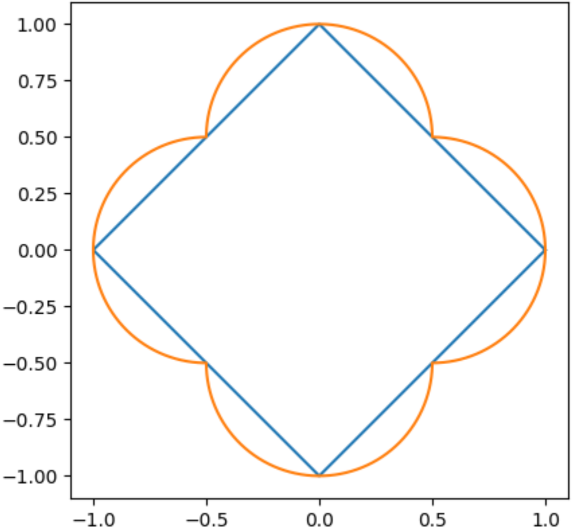
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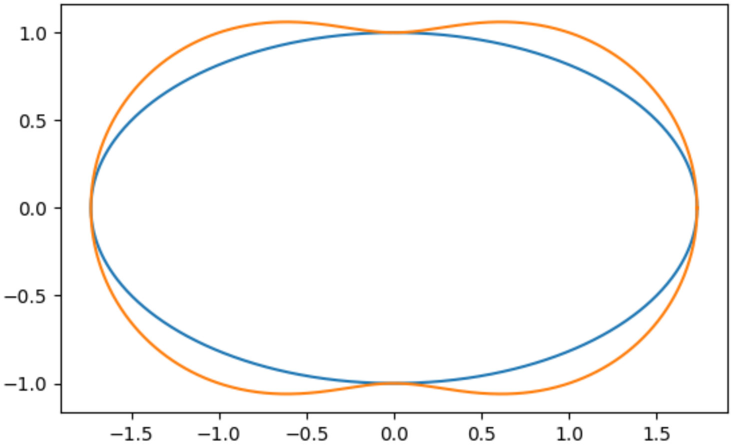
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Characterization of reciprocal bodies

Theorem 2

Given convex K, $0 \in K$, $\exists\, A$, s.t. $A' = K$ (i.e. K is reciprocal) iff K^* is convex.

So reciprocal bodies are "more convex" than others. I like to see it as *doubly* convex bodies. Let us provide an example of their use.

Let E be a subspace and Pr_E means the orthogonal projection on E.

Corollary

$(\mathrm{Pr}_E\, K)' = \mathrm{Pr}_E\, K'$ for reciprocal body K (and only such K-th).

Proof. Note $(\mathrm{Pr}_E\, K)^* = K^* \cap E$. (Check)

As K^* is convex, then

$$(\mathrm{Pr}_E K)' = \left((\mathrm{Pr}_E K)^*\right)^\circ = (K^* \cap E)^\circ = \mathrm{Pr}_E (K^*)^\circ = \mathrm{Pr}_E K'.$$

To present a sketch of the proof of the Theorem we need more operations.

Consider spherical inversion I

$$I(x) = x/|x|^2 \qquad x \neq 0 \in \mathbb{R}^n$$

and ϕ is a map on star-bodies in $\mathbb{R}^n$ s.t.

$$\phi(A) = \overline{\{Ix \mid x \notin A\}} \text{ – the closure of the image of } A^c.$$

On star-bodies ϕ is an involution and duality: if $r_A(\theta) = \sup\{\lambda > 0 : \lambda\theta \in A\}$ -radial function of A, then $r_{\phi A} = 1/r_A$.

More operator-type notation:

$$\phi : A \to \phi(A); \quad \circ\circ : A \to \operatorname{conv} A \quad (\circ\circ A := (A^\circ)^\circ = \operatorname{conv} A).$$

Note, operation ϕ does not preserve convexity. It maps a euclidean ball to balls (may be degenerated). In particular,

$$\phi(B_x) \text{ is (a low) half space (affine)}$$

perpendicular to the ray $\{\lambda x \mid \lambda \geq 0\}$ at $x/|x|^2$.

There follows

Fact 4. $\phi * = \circ$ (for any star body), i.e.

$$\phi(K^*) = K^\circ \quad \text{or} \quad K^* = \phi(K^\circ).$$

(Remember, ϕ is a "local" map.)

This implies that for convex bodies (put K° instead of K)

$$\phi * \circ = \mathrm{Id} \Longrightarrow * \circ = \phi.$$

Fact 4a.

$$* \circ = \phi \text{ on some } K \Longleftrightarrow K \text{ is convex.}$$

Corollary

Let $*K (= K^*)$ *convex (for convex* K*). Then* $* \circ * = \phi * = \circ$*.*
But $\circ * K = K'$*, i.e.* $(K')^* = K^\circ$*. Take* $\circ \Rightarrow$

$$K'' = (K')^{*\circ} = K.$$

So K^*-convex implies K is reciprocal ("easy" part of the theorem).

Some relations we have till now:

$$
\begin{aligned}
* &= \phi \circ \\
* \circ &= \phi && \text{(on convex bodies)} \\
\circ * &= {}' && (K^*)^\circ = K' \\
\phi * &= \circ && \phi(K^*) = K^\circ .
\end{aligned}
$$

Also $\circ \circ K = \operatorname{Conv} K$ and (as an example)

$$* \operatorname{Conv} * K = * \circ \circ * K = \phi \circ * K = \phi \circ \phi \phi * K = \phi \circ \phi \circ K = K^{**}.$$

Moreover, $* \operatorname{Conv} K = * K$ ($\forall K$-starbody).

And a Corollary of Fact 4:
For K-convex ($0 \in K$) ϕK is also convex iff K° is reciprocal.

Proof of the characterization theorem

Lemma

Let K be any convex body $0 \in K$. Consider the subset

$$T = \bigcup \{ B(x, |x|) \text{ and } B(x, |x|) \subset K \}$$

(so any such ball passes through 0 and is in K). Then T is a convex *subset of K.*

(Surprising! But that said – easy.)

Note that $T = \phi \circ \circ \phi \, K := \phi \, \mathrm{Conv} \, \phi K$.

(Formal checking: $\phi \, \partial B(x, |x|)$ is a hyperplane outside K^*.)

Using this lemma let us prove Theorem 2.

Proof.

We want to show that $K'' = K \Rightarrow K^*$ convex. This means $K'' := \circ * \circ * K = K$.

$$\begin{gathered}(\text{act by } *) \Rightarrow * \circ * \circ * K = * K \\ (\text{use } * = \phi \circ) * \circ \phi \circ \circ \phi \circ K = * K \\ * \circ [\phi \circ \circ \phi] \circ K = * K\end{gathered}$$

and $* \circ = \phi$ on convex sets, but, by the lemma, $\phi \circ \circ \phi(\circ K)$ convex,

$$\phi\phi \circ \circ \phi \circ K = * K \Rightarrow \circ \circ \phi \circ K = * K$$

which means $\text{Conv}(* K) = * K$ (recall $\phi \circ = *$). □

The above proof is not intuitive.

Let us see some intuition behind on one example.

Let $A \in D(T)$, i.e. $A' = T$. Then $T'' = T$ and $T' \in D(T)$. Recall T' is a maximal set in $D(T) : A \subset T'$. If $A \neq T'$, then it is not reciprocal (because otherwise $A = A'' = T'$).

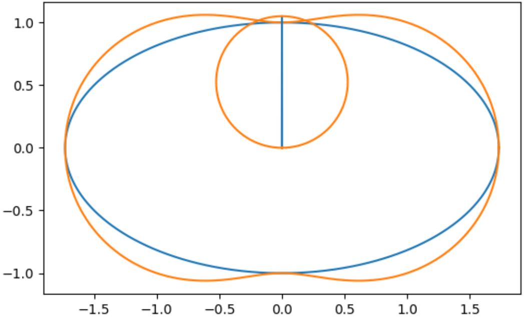
So, if K^* is not convex we would like to find another body K_1 s.t. $K \subsetneq K_1$ but $\operatorname{Conv} K_1^* = \operatorname{Conv} K^*$ and then $K_1' = K'$, i.e. K is not reciprocal.

Example: Our K is an ellipsoid E and $K_1 = \operatorname{Conv}(E \cup I)$, I is a special interval (see picture).

The last needed fact is

Fact 5. $[\operatorname{Conv}(K \cup T)]^* = K^* \cup T^*$

This fact and example demonstrate how lack of convexity of K^* is used to prove that K is not reciprocal.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Proof of the Lemma. Let $B_i = B_i(x_i, |x_i|) \subseteq K$, $i = 1, 2$.

Let $a_i \in B_i$. We should show that $\forall\, \lambda$, $0 < \lambda < 1$, $\exists$ a ball $B \subseteq K$ from our family of balls and $\lambda a_i + (1-\lambda)a_2 \in B$.

We will prove that $\forall\, z \in \mathrm{Conv}(B_1, B_2) := A$, $\exists$ such a ball $B \subset A(\subseteq K)$ and $z \in B$.

$$\text{Set} \qquad A = \bigcup_{\lambda \in [0,1]} \left\{(1-\lambda)B_1 + \lambda B_2\right\}$$
$$= \bigcup_{\lambda} B\big((1-\lambda)x_1 + \lambda x_2, (1-\lambda)|x_1| + \lambda |x_2|\big)$$

Then $\exists\, \lambda$ and $z \in B((1-\lambda)x_1 + \lambda x_2, (1-\lambda)|x_1| + \lambda|x_2|) = B^1$, and $0 \in B^1$ ball. Then $\exists$ a ball $\widetilde{B}$ inside this ball $B^1(\subseteq K)$ s.t. $0 \in \partial\widetilde{B}$, $z \in \widetilde{B}$.

More additions and an application

Facts:

1. Let B be a euclidean ball, $0 \in B$. Then B-indicatrix (of some ellipsoid) and B° is reciprocal. If 0 is not the center B then B is not reciprocal.

2. If K and T are reciprocal then $(K^\circ + T^\circ)^\circ$ is also reciprocal.

3. IF K and T are convex, such that $\phi(K)$ and $\phi(T)$ are convex, then $\phi(K+T)$ is also convex.