Karlsruhe Institute of Technology

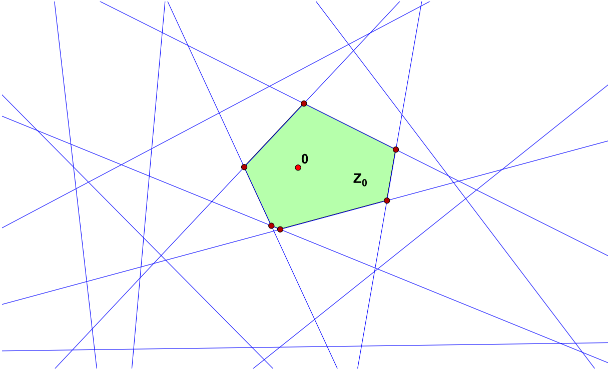
Spherical random tessellations and analytic convexity

Daniel Hug | September 2018

CASTRO URDIALES

Poisson line tessellation

- Poisson line process in $\mathbb{R}^2$, stationary and isotropic
- Stationary, isotropic line tessellation: random infinite collection of polygonal cells
- **Crofton cell or zero cell** Z_0: containing the origin



Kendall's Conjecture (1940s, 1987)

"The conditional law for the shape of Z_0, given the area $A(Z_0)$ of Z_0, converges weakly, as $A(Z_0) \to \infty$, to the degenerate law concentrated at the spherical shape."

- R. Miles (1995)
- I. N. Kovalenko (1997, 1999)
- D. Hug, M. Reitzner, R. Schneider (2004)
- D. Hug, R. Schneider (2007), ...
- Calka (2010, '13 (surveys), ...)
- G. Bonnet (2016)
- ...

Poisson hyperplane tessellation in $\mathbb{R}^d$

Consider a **Poisson hyperplane process**

$$X = \{H_i : i \in \mathbb{N}\} = \sum_{i \in \mathbb{N}} \delta_{H_i}$$

with $H_i \in \mathbf{A}(d, d-1)$, which is **stationary and isotropic**.

The **intensity measure** of X is a measure on $\mathbf{A}(d, d-1)$ given by

$$\mathbb{E}X(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \mathbf{1}\{u^\perp + tu \in \cdot\}\, dt\, \sigma_0(du).$$

Here σ_0 is normalized $\mathcal{H}^{d-1}$, $\gamma > 0$ is the intensity of X.

Let $\mathcal{H}_K := \{H \in \mathbf{A}(d, d-1) : H \cap K \neq \emptyset\}$. The Poisson assumption means that $X(\mathcal{H}_K)$ is Poisson distributed with mean value $\mathbb{E}X(\mathcal{H}_K)$.

The **hitting functional** of X is

$$K \mapsto \mathbb{E}X(\mathcal{H}_K) \sim V_1(K) \quad \text{for } K \in \mathcal{K}^d,$$

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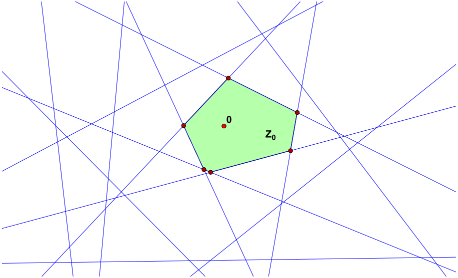

Let Z_0 be the zero cell/Crofton cell of the tessellation induced by X.

What is the limit shape of Z_0 – if it exists – given $V_d(Z_0) \to \infty$?

Does the shape of Z_0 concentrate at a particular (class of) shape(s) given $V_d(Z_0) \to \infty$?

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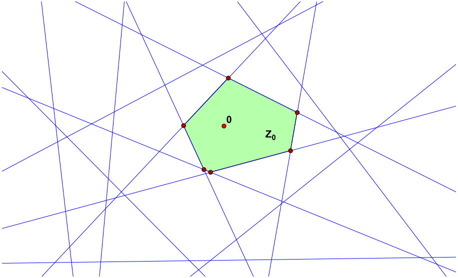

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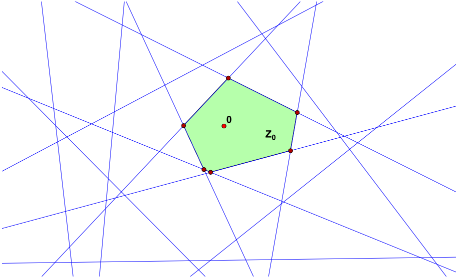

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A deviation result

based on a **deviation functional**

$\vartheta(Z_0) =$ "scaling, translation, rotation invariant distance of Z_0 from B^d".

Theorem (Hug, Reitzner, Schneider (2004), a special case **. . .)**

If X is stationary and isotropic in $\mathbb{R}^d$, $\varepsilon \in (0,1)$, and $a^{1/d}\gamma \geq 1$, then

$$\mathbb{P}\left(\vartheta(Z_0) \geq \varepsilon \mid V_d(Z_0) \geq a\right) \leq c \exp\left(-c_1\, \varepsilon^{d+1} a^{1/d} \gamma\right),$$

where $c = c(d, \varepsilon)$ and $c_1 = c_1(d)$.

Extensions (with Rolf Schneider): no isotropy assumption, relaxed stationarity assumption, typical cells, Voronoi and Delaunay tessellations, lower-dimensional weighted typical faces, various other size functionals, axiomatic approach, asymptotic distributions

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Asymptotic distribution

Recall: $V_1(K)$ denotes the mean width of K.

Theorem (Hug, Schneider (2007), a special case . . .)

$$\lim_{a\to\infty} a^{-1/d} \ln \mathbb{P}(V_d(Z_0) \geq a) = -\tau\,\gamma,$$

where

$$\tau \sim \min\{V_1(K) : V_d(K) = 1\}.$$

Some ingrediens:

- Polytopal approximation with few vertices
- Separate treatment of elongated cells
- Use of homogeneity arguments
- **Isoperimetric and stability problems!**

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Isoperimetry and stability

Urysohn inequality:

$$V_1(K) \geq c(d)\, V_d(K)^{1/d}.$$

Equality holds if and only if K is a ball.

Quantitative stability improvement:

$$V_1(K) \geq \left(1 + a(d)\, \vartheta(K)^{d+1}\right) c(d)\, V_d(K)^{1/d}.$$

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Which cells arise?

For isotropic tessellations, the following assumptions are always satisfied:

(A): The support of the directional distribution φ of X is $\mathbb{S}^{d-1}$.

(B): φ is zero on each great subsphere of $\mathbb{S}^{d-1}$.

Theorem (Reitzner & Schneider)

Let X be a stationary Poisson hyperplane tessellation in $\mathbb{R}^d$ with the property that φ satisfies (A) and (B). Then a.s. the set of translates of the cells of X is dense in $\mathcal{K}$.

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Which cells arise how often?

The cells in a stationary Poisson hyperplane tessellation are a.s. simple polytopes.

Under (A) and (B) no other restrictions arise. The following improves a result by Reitzner & Schneider.

Theorem (Schneider)

Let X be a stationary Poisson hyperplane tessellation in $\mathbb{R}^d$. Suppose φ satisfies (A) and (B). Then, with probability one, every combinatorial type of a simple d-polytope appears in X with positive density.

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- Spherical tessellations
- Large cells?
- Geometric inequalities
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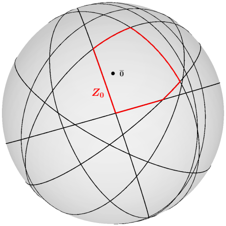
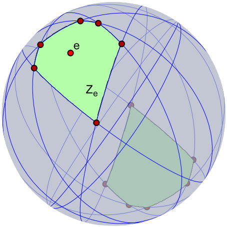
Spherical tessellations by great subspheres

- Let X be an isotropic Poisson point process in $\mathbb{S}^d \subset \mathbb{R}^{d+1}$
- Spherical isotropic Poisson process of great subspheres

$$\widetilde{X} := \{x^\perp \cap \mathbb{S}^d : \; x \in X\}$$

- Crofton cell Z_0





Intensity measure and hitting functional

- Spherically convex bodies: $\mathcal{K}_s^d \ni K$
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$$\mathbb{E}\widetilde{X}(\mathcal{H}_K) = \gamma_S \int_{\mathbb{S}^d} \mathbf{1}\{x^\perp \cap K \neq \emptyset\}\, \mathcal{H}^d(dx)$$

$$U_1(K) := (2\beta_d)^{-1} \int_{\mathbb{S}^d} \mathbf{1}\{x^\perp \cap K \neq \emptyset\}\, \mathcal{H}^d(dx)$$

- Void probability

$$\mathbb{P}(\widetilde{X}(\mathcal{H}_K) = 0) = \exp\left(-2\gamma_S \beta_d U_1(K)\right)$$

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A spherical Urysohn inequality

Theorem (Gao, Hug, Schneider (2003))

Let $K \in \mathcal{K}_s^d$ and let $C \subset \mathbb{S}^d$ be a spherical cap with $\mathcal{H}^d(C) = \mathcal{H}^d(K)$. Then

$$U_1(K) \geq U_1(C).$$

Equality holds if and only if K is a spherical cap.

We need a quantitative improvement / stability result!

Is K close to C (in a quantitative way), if $U_1(K)$ is ε-close to $U(C)$?

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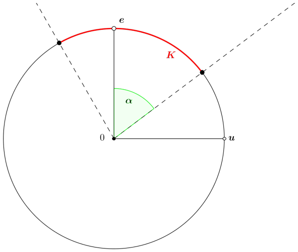

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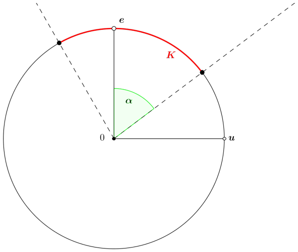

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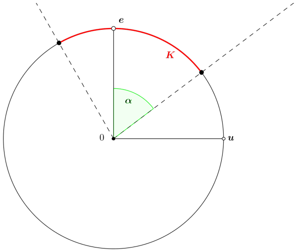

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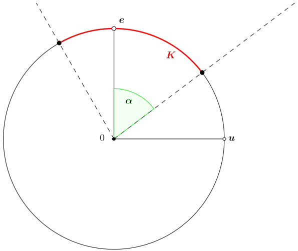

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$$\Delta(K) := \inf \left\{ \| D \circ \alpha_{K,e} - \overline{D \circ \alpha_{K,e}} \|_{L^2(S_e)} : e \in K \cap (-K^*) \right\}.$$

A geometric stability result

Theorem (Hug, Reichenbacher)

Let $K \in \overline{\mathcal{K}}_s^d$ and let C be a spherical cap with $\mathcal{H}^d(K) = \mathcal{H}^d(C) > 0$. Let $\alpha_0 \in (0, \pi/2)$ be such that $\alpha_0 \leq \alpha_C$. Then

$$U_1(K) \geq (1 + \widetilde{\gamma}\,\Delta(K)^2)U_1(C)$$

with

$$\widetilde{\gamma} = 2 \cdot \min\left\{ \frac{\binom{d+1}{2} \sin^{d+1}(\alpha_0) \tan^{-2d}(\alpha_C)}{d + d\binom{d+1}{2}\left(\frac{\pi}{2}\right)^2 \tan^{-d}(\alpha_C)}, \left(\frac{2}{\pi}\right)^2 D\left(\frac{\pi}{2} - \alpha_C\right) \right\}.$$

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A deviation result for the spherical Crofton cell

Theorem (Hug, Reichenbacher)

Let $0 < a < \beta_d/2$ *and* $0 < \varepsilon < 1$. *Then there are constants* $\widetilde{c}_1, \widetilde{c}_2 > 0$ *such that*

$$\mathbb{P}(\Delta(Z_0) \geq \varepsilon \mid \mathcal{H}^d(Z_0) \geq a) \leq \widetilde{c}_1 \cdot \exp\left(-\widetilde{c}_2 \cdot \varepsilon^{2(d+1)} \cdot \gamma_S\right),$$

where $\widetilde{c}_1 = \widetilde{c}_1(a, \varepsilon, d)$, $\widetilde{c}_2 = \widetilde{c}_2(a, d)$.

Asymptotic distribution

Theorem (Hug, Reichenbacher)

Let $0 < a < \beta_d/2$. *Then*

$$\lim_{\gamma_S \to \infty} \gamma_S^{-1} \cdot \ln \mathbb{P}(\mathcal{H}^d(Z_0) \geq a) = -2\beta_d \cdot U_1(B_a),$$

where B_a *is a spherical cap of volume a.*

Similar results have been obtained for binomial processes and for the spherical inradius as the size functional, but also for general continuous, increasing size functionals $\Sigma \neq 0$ vanishing on one-pointed sets.

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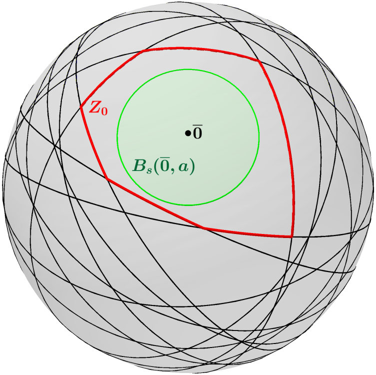
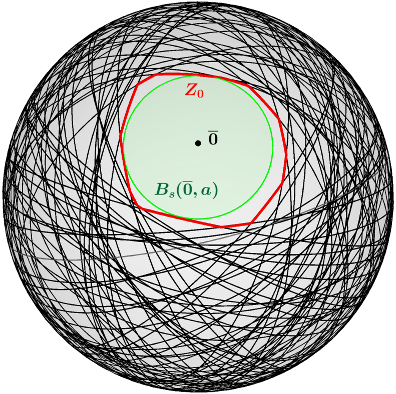
Illustration



$\gamma_S = 1$ (17 great subspheres)

Illustration



$\gamma_S = 10$ (118 great subspheres)

Spherical Poisson–Voronoi cells

Let X be an isotropic Poisson process on $\mathbb{S}^d$ with intensity γ_s, and let $X' = \{C(x, X) : x \in X\}$ be the associated Poisson–Voronoi tessellation.

The distribution of the typical cell Z then satisfies

$$\mathbb{P}(Z \in \cdot) = \mathbb{P}(C(\bar{o}, X + \delta_{\bar{o}}) \in \cdot).$$

Hitting and deviation functional

Hence Z is equal in distribution to the Crofton cell of a (non-isotropic) Poisson process Y of great subspheres with hitting functional

$$\mathbb{E}Y(\mathcal{H}_K) = \gamma_s \widetilde{U}(K), \qquad \bar{o} \in K \in \mathcal{K}_s^d,$$

where

$$\widetilde{U}(K) = 2 \int_{\bar{o}^\perp \cap \mathbb{S}^d} \int_{A_s(u)} \sin^{d-1}\left(2d_s(\tilde{S}_u, t)\right) \mathbf{1}\{t^\perp \cap K \neq \emptyset\} \, \mathcal{H}^1(dt) \, \mathcal{H}^{d-1}(du)$$

with $\tilde{S}_u = \{-\bar{o}, u\}$ and $A_s(u) = \operatorname{arc}(-\bar{o}, u)$.

Define

$$\begin{aligned} r_s(K) &:= \max\{r \geq 0 : B_s(\bar{o}, r) \subset K\} \\ R_s(K) &:= \min\{r \geq 0 : B_s(\bar{o}, r) \supset K\} \\ \vartheta(K) &:= R_s(K) - r_s(K). \end{aligned}$$

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Geometric stability

Theorem (Hug, Reichenbacher)

Let $a \in (0, \pi/2)$, $\bar{o} \in K \in \mathcal{K}_s^d$ *with* $r_s(K) \geq a$ *and* $C := B_s(\bar{o}, a)$. *Then*

$$\widetilde{U}(K) \geq \widetilde{U}(C) = \mathcal{H}^d(B_s(\bar{o}, 2a)).$$

Equality holds if and only if $K = C$.

More generally,

$$\widetilde{U}(K) \geq \left(1 + c_5(a, d)\, \vartheta(K)^d\right)\, \widetilde{U}(C).$$

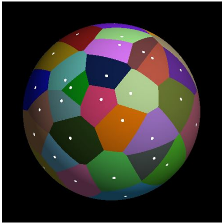
Shape deviation

Theorem (Hug, Reichenbacher)

Let $a \in (0, \pi/2)$ and $\varepsilon \in (0, 1]$. Let Z be the typical cell of the Voronoi tessellation associated with an isotropic Poisson point process with intensity γ_s on $\mathbb{S}^d$. Then

$$\mathbb{P}(R_s(Z) - r_s(Z) \geq \varepsilon \mid r_s(Z) \geq a) \leq c_6 \cdot \exp\left(-c_7 \cdot \varepsilon^d \cdot \gamma_S\right),$$

where $c_6 = c_6(a, d, \varepsilon)$ and $c_7 = c_7(a, d)$.

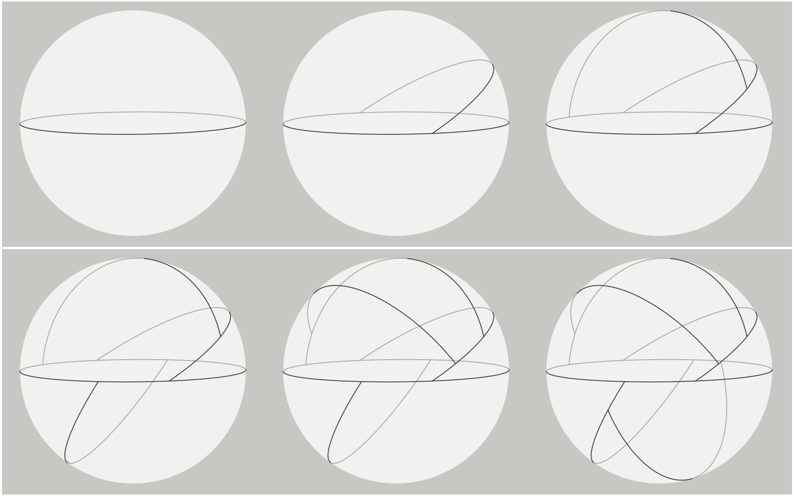


Davies, J. https://www.jasondavies.com/maps/voronoi

Splitting tessellations in spherical space

Joint work with Christoph Thäle

A recursive cell splitting scheme:



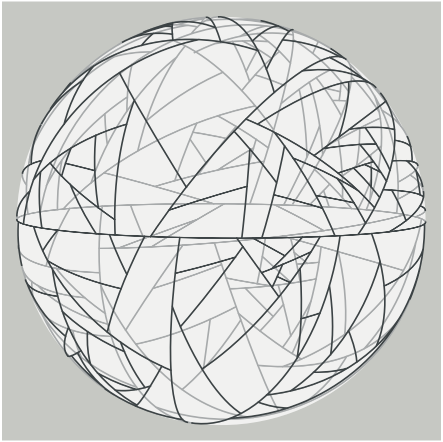


Figure: Illustration of a splitting tessellation.

A splitting process via cell-splitting

Define $\oslash : \mathbb{P}^d \times \mathbb{S}_{d-1} \times \mathbb{T}^d \to \mathbb{T}^d$ by

$$\oslash(c, S, T) := (T \setminus \{c\}) \cup \{c \cap S^+, c \cap S^-\} \in \mathbb{T}^d ,$$

if $c \in T$, $S \in \mathbb{S}_{d-1}[c]$, and where $S^\pm$ are the two closed hemispheres determined by S; otherwise $\oslash(c, S, T) := T$.

Definition

A splitting process $(Y_t)_{t\geq 0}$ with initial tessellation $Y_0 := \{\mathbb{S}^d\}$ is a continuous time, pure jump Markov process on $\mathbb{T}^d$ with generator

$$(\mathcal{A}f)(T) := \sum_{c\in T} \int_{\mathbb{S}_{d-1}[c]} \left[f(\oslash(c, S, T)) - f(T)\right] \nu_{d-1}(\mathrm{d}S) , \qquad T \in \mathbb{T}^d ,$$

where $f \in \mathcal{F}_b(\mathbb{T}^d)$. For $t > 0$ we call Y_t a splitting tessellation.

Note that the **unbounded** intensity function λ of $Y, \mathcal{A}$ is $\lambda(T) = |T|$, $T \in \mathbb{T}^d$.

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An auxiliary martingale

Lemma

Let E be a Borel space and let $(X_t)_{t\geq 0}$ be a Markov process with values in E and with generator $\mathcal{L}$ whose domain is $D(\mathcal{L})$. Then, for $f \in D(\mathcal{L})$, the random process

$$f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)\,\mathrm{d}s\,, \qquad t \geq 0\,,$$

is a martingale with respect to the filtration induced by $(X_t)_{t\geq 0}$. If $(X_t)_{t\geq 0}$ is a jump process with bounded intensity function, then $\mathcal{F}_b(E) = D(\mathcal{L})$.

Applications

Proposition

Let $\phi : \mathbb{P}^d \to \mathbb{R}$ *be bounded and measurable. Define*

$$\Sigma_\phi(T) := \sum_{c \in T} \phi(c) = \int_{\mathbb{P}^d} \phi \, \mathrm{d}\mu_T \,, \qquad T \in \mathbb{T}^d \,.$$

Then the stochastic process

$$M_t(\phi) := \Sigma_\phi(Y_t) - \Sigma_\phi(Y_0) - \int_0^t (\mathcal{A}\Sigma_\phi)(Y_s) \, \mathrm{d}s \,, \qquad t \geq 0 \,,$$

is a martingale with respect to $\mathcal{Y}$*, the filtration generated by* $(Y_t)_{t \geq 0}$*.*

Proposition

Let $\phi_i : \mathbb{P}^d \to \mathbb{R}$ *for* $i \in \{1, 2\}$ *be bounded and measurable. Define*

$$\Sigma_{\phi_1,\phi_2}(T) := \Sigma_{\phi_1}(T)\,\Sigma_{\phi_2}(T)\,, \qquad T \in \mathbb{T}^d\,.$$

Then the stochastic process

$$M_t(\phi_1, \phi_2) := \Sigma_{\phi_1,\phi_2}(Y_t) - \Sigma_{\phi_1,\phi_2}(Y_0) - \int_0^t (\mathcal{A}\Sigma_{\phi_1,\phi_2})(Y_s)\,\mathrm{d}s\,, \qquad t \geq 0\,,$$

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By time augmentation, we can also treat functionals of the form

$$\Psi_{\phi_1,\phi_2}(T, t) := (\Sigma_{\phi_1}(T) - b_1 t^{\nu_1})(\Sigma_{\phi_2}(T) - b_2 t^{\nu_2})\,, \qquad T \in \mathbb{T}^d, t \geq 0\,.$$

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Expected spherical curvature measures

For $t \geq 0$, $j \in \{0, \ldots, d\}$ and $A \in \mathcal{B}(\mathbb{S}^d)$, define

$$\Sigma_j(t; A) := \sum_{c \in Y_t} \phi_j(c, A) \, .$$

More generally, if $h : \mathbb{S}^d \to \mathbb{R}$ is bounded, measurable and μ is a finite Borel measure on $\mathbb{S}^d$, we write $\mu(h) := \int_{\mathbb{S}^d} h \, \mathrm{d}\mu$.

Theorem

Let $t \geq 0$ and $j \in \{0, \ldots, d\}$. Then

$$\mathbf{E}\Sigma_j(t; h) = \frac{t^{d-j}}{(d-j)!} \frac{\mathcal{H}^d(h)}{\beta_d} \, ,$$

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Idea of proof

The random process

$$\Sigma_j(t;h) - \int_0^t \sum_{c \in Y_s} \int_{\mathbb{S}_{d-1}[c]} [\phi_j(c \cap S^+, h) + \phi_j(c \cap S^-, h) - \phi_j(c, h)] \, \nu_{d-1}(\mathrm{d}S) \, \mathrm{d}s$$

is a $\mathcal{Y}$-martingale. The **valuation property** of ϕ_j yields that

$$\phi_j(c \cap S^+, h) + \phi_j(c \cap S^-, h) - \phi_j(c, h) = \phi_j(c \cap S, h) \, .$$

Taking expectations and applying the **local spherical Crofton formula**,

$$\mathbf{E}\Sigma_j(t;h) = \mathbf{E} \int_0^t \sum_{c \in Y_s} \int_{\mathbb{S}_{d-1}[c]} \phi_j(c \cap S, h) \, \nu_{d-1}(\mathrm{d}S) \, \mathrm{d}s$$

$$= \mathbf{E} \int_0^t \sum_{c \in Y_s} \phi_{j+1}(c, h) \, \mathrm{d}s = \mathbf{E} \int_0^t \Sigma_{j+1}(s;h) \, \mathrm{d}s \, .$$

Now work recursively and use that, with probability one,

$$\Sigma_d(s;h) = \sum_{c \in Y_s} \phi_d(c, h) = \sum_{c \in Y_s} \frac{\mathcal{H}^d(h \mathbf{1}_c)}{\beta_d} = \frac{\mathcal{H}^d(h)}{\beta_d} \, .$$

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Idea of proof

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$$\Sigma_j(t;h) - \int_0^t \sum_{c\in Y_s} \int_{\mathbb{S}_{d-1}[c]} [\phi_j(c\cap S^+,h) + \phi_j(c\cap S^-,h) - \phi_j(c,h)]\,\nu_{d-1}(\mathrm{d}S)\,\mathrm{d}s$$

is a $\mathcal{Y}$-martingale. The **valuation property** of ϕ_j yields that

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Now work recursively and use that, with probability one,

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Theorem

If $t \geq 0$ and $h : \mathbb{S}^d \to \mathbb{R}$ is bounded and measurable, then

$$\operatorname{Var} \Sigma_{d-1}(t;h) = \frac{\pi \beta_{d-2}}{\beta_d \beta_{d-1}^2} \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \frac{1 - \exp\left(-\frac{1}{\pi}\ell(x,y)t\right)}{\ell(x,y)\sin(\ell(x,y))} \times h(x)h(y)\,\mathcal{H}^d(\mathrm{d}x)\,\mathcal{H}^d(\mathrm{d}y) < \infty.$$

- Proof uses auxiliary martingales and basic spherical integral geometry.
- Covariances and variances for different functions h and lower order curvature measures can also be determined.
- The mean and variance of the Hausdorff measure of the boundary Z_t of Y_t can be obtained as a special case.
- Euclidean analogue is due to Schreiber & Thäle.

Next we study an application.

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Spherical K-function and pair-correlation function

Definition

Let $\mathbf{M}$ be an isotropic random measure on $\mathbb{S}^d$ with intensity $\mu \in (0, \infty)$, determined by $\mathbf{E}[\mathbf{M}(\,\cdot\,)] = \mu \beta_d^{-1} \mathcal{H}^d(\,\cdot\,)$ on $\mathbb{S}^d$.

The spherical K-function of $\mathbf{M}$ can be defined by

$$K_{\mathbf{M}}(r) := \frac{1}{\mu^2} \mathbf{E} \int_{(\mathbb{S}^d)^2} \mathbf{1}(\ell(x, y) \leq r)\, \mathbf{M}^2(\mathrm{d}(x, y))$$

where $B(e, r) = \{x \in \mathbb{S}^d : \ell(e, x) \leq r\}$.

If $K_{\mathbf{M}}$ is differentiable, then

$$g_{\mathbf{M}}(r) := \frac{\beta_d}{\beta_{d-1} (\sin r)^{d-1}} K'_{\mathbf{M}}(r)\,, \qquad r \in (0, \pi)\,,$$

is the spherical pair-correlation function of $\mathbf{M}$.

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K-function and g-function for specific M

Choose the random measure $\mathbf{M} = \mathcal{H}^{d-1} \llcorner Z_t$.

Theorem

If $t > 0$ and $r \in (0, \pi)$, then

$$K_{d,t}(r) = \frac{\beta_{d-1}}{\beta_d} \int_0^r \left(1 + \pi \frac{\beta_{d-2}\beta_d}{\beta_{d-1}^2} \frac{1 - \exp\left(-\frac{t}{\pi}\varphi\right)}{t^2 \varphi \sin\varphi}\right) (\sin\varphi)^{d-1} \, \mathrm{d}\varphi$$

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We compare this to Poisson hypersphere tessellations.

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Poisson hypersphere tessellation

Let η_t be a Poisson process on $\mathbb{S}^d$ with intensity measure $t\,\beta_d^{-1}\mathcal{H}^d$. Denote by $\overline{Y}_t$ the tessellation of $\mathbb{S}^d$ induced by η_t, and let

$$\overline{Z}_t := \bigcup_{u \in \eta_t} (u^\perp \cap \mathbb{S}^d)$$

be the associated random closed set.

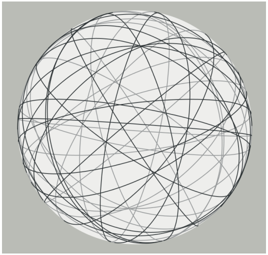

Figure: Illustration of Poisson circle tessellation on $\mathbb{S}^2$.

K-function and g-function for specific M

The random measure $\mathcal{H}^{d-1} \llcorner \overline{Z}_t$ is isotropic and its intensity $\overline{\mu} := \mathbf{E}\mathcal{H}^{d-1}(\overline{Z}_t \cap \mathbb{S}^d)$ equals $\overline{\mu} = t\beta_{d-1}$.

This is also the intensity of $\mathcal{H}^{d-1} \llcorner Z_t$.

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$$K_{d,t} \leq \overline{K}_{d,t}.$$

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Illustration

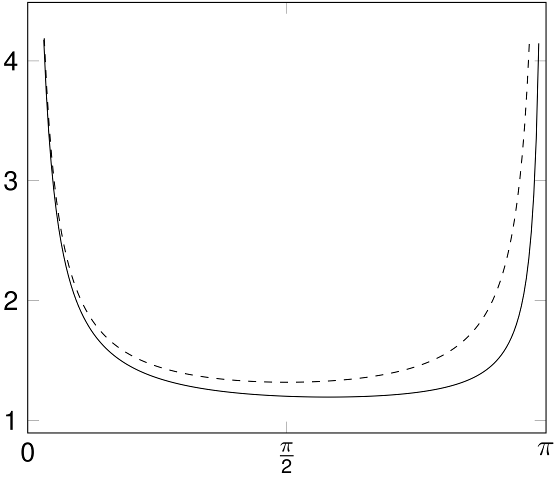

Figure: The spherical pair-correlation functions $g_{2,2}(r)$ (solid curve) and $\overline{g}_{2,2}(r)$ (dashed curve).

Dynamic description of Poisson tessellation process

Splitting tessellations and Poisson hypersphere tessellations are linked to each other: for $T \in \mathbb{T}^d$ and $S \in \mathbb{S}_{d-1}$, we define

$$\otimes(S,T) := (T \setminus \{c \in T : \mathrm{int}(c) \cap S \neq \emptyset\}) \cup \bigcup_{\substack{c \in T \\ \mathrm{int}(c) \cap S \neq \emptyset}} \{c \cap S^+, c \cap S^-\} \in \mathbb{T}^d.$$

Define a continuous-time Markov process $(\overline{Y}_t)_{t \geq 0}$ with initial tessellation $\overline{Y}_0 = \{\mathbb{S}^d\}$ in $\mathbb{T}^d$ via its generator $\overline{\mathcal{A}}$, where

$$(\overline{\mathcal{A}}f)(T) = \int_{\mathbb{S}_{d-1}} [f(\otimes(S,T)) - f(T)]\, \nu_{d-1}(\mathrm{d}S), \qquad T \in \mathbb{T}^d,$$

where $f \in \mathcal{F}_b(\mathbb{T}^d)$.

For $t > 0$, the random tessellation $\overline{Y}_t$ has the same distribution as a Poisson hypersphere tessellation with intensity t.

Dynamic description of Poisson tessellation process

Splitting tessellations and Poisson hypersphere tessellations are linked to each other: for $T \in \mathbb{T}^d$ and $S \in \mathbb{S}_{d-1}$, we define

$$\otimes(S, T) := (T \setminus \{c \in T : \mathrm{int}(c) \cap S \neq \emptyset\}) \cup \bigcup_{\substack{c \in T \\ \mathrm{int}(c) \cap S \neq \emptyset}} \{c \cap S^+, c \cap S^-\} \in \mathbb{T}^d.$$

Define a continuous-time Markov process $(\overline{Y}_t)_{t \geq 0}$ with initial tessellation $\overline{Y}_0 = \{\mathbb{S}^d\}$ in $\mathbb{T}^d$ via its generator $\overline{\mathcal{A}}$, where

$$(\overline{\mathcal{A}} f)(T) = \int_{\mathbb{S}_{d-1}} [f(\otimes(S, T)) - f(T)] \, \nu_{d-1}(\mathrm{d}S), \qquad T \in \mathbb{T}^d,$$

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Relationships for intensity measures

Consider the random measure $\mathcal{M}_t$ and its intensity measure $\mathbb{M}_t$ on $\mathbb{P}^d$,

$$\mathcal{M}_t := \sum_{c \in Y_t} \delta_c \qquad \text{and} \qquad \mathbb{M}_t := \mathbf{E}\mathcal{M}_t, \quad t \geq 0.$$

Similarly, for a Poisson hypersphere tessellation $\overline{Y}_t$ with intensity $t \geq 0$,

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Theorem

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Sketch of proof

Let $\phi : \mathbb{P}^d \to \mathbb{R}$ be bounded and measurable. Then

$$\Sigma_\phi(Y_t) - \Sigma_\phi(Y_0) - \int_0^t \sum_{c \in Y_s} \int_{\mathbb{S}_{d-1}[c]} [\phi(c \cap S^+) + \phi(c \cap S^-) - \phi(c)] \, \nu_{d-1}(\mathrm{d}S) \, \mathrm{d}s$$

is a $\mathcal{Y}$-martingale. Take expectations, we get

$$\int \phi(c) \, \mathbb{M}_t(\mathrm{d}c) = \phi(\mathbb{S}^d) + \int_0^t \int \int_{\mathbb{S}_{d-1}[c]} [\phi(c \cap S^+) + \phi(c \cap S^-) - \phi(c)] \, \nu_{d-1}(\mathrm{d}S) \, \mathbb{M}_s(\mathrm{d}c) \, \mathrm{d}s \, .$$

Sketch of proof

Denote by $\mathbb{M}_{\mathrm{bv}}(\mathbb{P}^d)$ the Banach space of real-valued Borel measures on $\mathbb{P}^d$ with the total variation norm $\|\cdot\|_{\mathrm{TV}}$. Then the linear operator

$$\Gamma : \mathbb{M}_{\mathrm{bv}}(\mathbb{P}^d) \to \mathbb{M}_{\mathrm{bv}}(\mathbb{P}^d),\ \mu \mapsto \int\int_{\mathbb{S}_{d-1}} [\delta_{c\cap S^+} + \delta_{c\cap S^-} - \delta_c]\, \nu_{d-1}(\mathrm{d}S)\, \mu(\mathrm{d}c),$$

is bounded with operator norm $\|\Gamma\| \leq 3$.

It follows that

$$\mathbb{M}_t = \delta_{\mathbb{S}^d} + \int_0^t \Gamma(\mathbb{M}_s)\, \mathrm{d}s\,, \qquad t \geq 0\,,$$

in $\mathbb{M}_{\mathrm{bv}}(\mathbb{P}^d)$ and $\|\mathbb{M}_t - \mathbb{M}_r\|_{\mathrm{TV}} \leq 3c_a|t-r|$ for $0 \leq r \leq t \leq a$.

By similar arguments it can be shown that $\overline{\mathbb{M}}_t$, $t \geq 0$, satisfies the same initial value problem.

Since the solution is unique, the result follows.

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Conference on Geometry and Probability

Subject: Convex, Discrete and Stochastic Geometry

Date: September 6 - 11, 2020

Venue: Bad Herrenalb (near Karlsruhe) in the Black Forest

Note: Rolf Schneider (March 1940) and Wolfgang Weil (April 1945)