Extensions and non-extensions of the Weyl principle

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Castro Urdiales, September 2018

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- **Curvature integrals.** If ∂K is C^2 with principal curvatures $(\kappa_j)_{j=1}^{n-1}$

$$\mu_k(K) = c_{n,k} \int_{\partial K} \sigma_{n-1-k}(\kappa_1, \dots, \kappa_{n-1}) d\,\mathrm{vol}_{n-1}$$

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There is an obvious globalization map $glob : \mathcal{C}^\infty(M) \to \mathcal{V}^\infty(M)$.

Filtration by homogeneity

Theorem (Alesker)

There is a canonic filtration

$$\mathcal{V}^\infty(M) = \mathcal{W}_0(M) \supset \mathcal{W}_1(M) \supset \cdots \supset \mathcal{W}_n(M)$$

such that $\mathcal{W}_k(M)/\mathcal{W}_{k+1}(M) = C^\infty(M, \mathrm{Val}_k(TM))$.

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A similar filtration $\mathcal{C}_k^\infty(M)$ can be defined on the curvature measures (Solanes-Wannerer).

Riemannian Lipschitz-Killing curvatures

Theorem (Weyl)

Let (M^n, g) be a closed Riemannian manifold. Embed isometrically $M \subset \mathbb{R}^N$. Let M_ϵ denote the ϵ-extension. Then $\mathrm{vol}_N(M_\epsilon) = \sum_{k=0}^n \omega_{n-k}\mu_k(M)\epsilon^{N-k}$ for small ϵ. Remarkably, μ_k only depend on (M, g).

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Let (M^n, g) be a Riemannian manifold. There is a canonic collection of valuations $\{\mu_k\}_{k=0}^n \in \mathcal{V}^\infty(M)$ (curvature measures $\{LK_k\}_{k=0}^n$) that can be obtained by fixing an isometric embedding $M \subset \mathbb{R}^N$ and restricting the intrinsic volumes of $\mathbb{R}^N$ (resp. Federer curvature measures) to M. Those valuations (curvature measures) can be described intrinsically through the curvature tensor.

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Examples: For $X \subset M$, $\mu_0(X) = \chi$, $\mu_n(X) = \mathrm{vol}_n(X)$, $\mu_{n-1}(X) = \frac{1}{2}\mathrm{vol}_{n-1}(\partial X)$.

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2. There is a (canonically defined, explicit) $\omega \in \Omega^{n-1}(SM)$ s.t. for $X \in \mathcal{P}(M^n)$, $\chi(X) = (2\pi)^{-n/2}\int_X \mathrm{Pfaff}(\Omega) + \int_{NX}\omega$.

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Cartan apparatus. On the bundle of orthonormal frames $\Phi M = \{(x, E_0, \ldots, E_n)\}$ over M^{n+1}, there are:

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Definition: The (normalized) L-K curvature measures are

$$LK_k = \frac{\pi^k}{k!\omega_k} \sum_{j=0}^\infty \binom{\frac{k}{2}+j}{j} 4^{-j} C_{k+2j,j} \in \mathcal{C}_k^\infty(M)$$

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Theorem (Fu-Wannerer '17)

The Lipschitz-Killing curvatures are the unique universally defined curvature measures on Riemannian manifolds that are invariant to isometric embeddings.

Pseudo-Riemannian crash course

Let M^n be a manifold equipped with a smooth field Q of non-degenerate quadratic forms of (necessarily constant) signature (p, q), e.g.
$ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$.

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Theorem (Nash)

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Theorem (Chern '62, Avez '62)

The Chern-Gauss-Bonnet theorem holds for pseudo-Riemannian manifolds. Namely, for closed, oriented, even-dimensional $M^{p,q}$,

$$\chi(M) = (2\pi)^{-(p+q)/2} \int_M \mathrm{Pfaff}(\tilde{\Omega})$$

Here $\tilde{\Omega}_{ij} = \Omega_{ij}\epsilon_j$, $\epsilon_1 = \cdots = \epsilon_p = 1$, $\epsilon_{p+1} = \cdots = \epsilon_n = -1$.

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Remark

This appears to be the reason why Chern and Avez only obtain the interior term. Later works by Birman-Nomizu ('83) and Gilkey-Park ('14) also contain a boundary term with the assumption that the boundary of the subset has non-degenerate metric. That is, if the boundary is connected, it necessarily has fixed signature.

Generalized valuations on manifolds

Roughly speaking, a generalized valuation $\phi \in \mathcal{V}^{-\infty}(M)$ is a functional on sufficiently nice subsets $X \in \mathcal{P}(M)$ given by $\phi(X) = \int_X \mu + \int_{N^*X} \omega$ for some currents (distributional forms) $\mu \in \mathcal{D}_0(M)$, $\omega \in \mathcal{D}_n(S^*M)$.

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Example

Given $A \in \mathcal{P}(M)$, $\chi_A := \chi(\bullet \cap A)$ is a generalized valuation. Such valuations are the building blocks of Crofton formulas.

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Roughly speaking, a generalized valuation $\phi \in \mathcal{V}^{-\infty}(M)$ is a functional on sufficiently nice subsets $X \in \mathcal{P}(M)$ given by $\phi(X) = \int_X \mu + \int_{N^*X} \omega$ for some currents (distributional forms) $\mu \in \mathcal{D}_0(M)$, $\omega \in \mathcal{D}_n(S^*M)$.

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- The space $\mathcal{C}^{-\infty}(M)$ of generalized curvature measures can be defined similarly. $\Phi \in \mathcal{C}^{-\infty}(M)$ is then a valuation on sufficiently nice $X \in \mathcal{P}(M)$, with values in $\mathcal{M}^{-\infty}(M)$.

Intrinsic volumes of $\mathbb{R}^{p,q}$

Assume $p, q \geq 1$, $n = p + q$. Write $Q(x) = x_1^2 + \cdots + x_p^2 - \cdots - x_n^2$.

Theorem (Alesker-F. '13, Bernig-F. '16)

In $\mathbb{R}^{p,q}$, the space of $\mathrm{O}(p,q)$-invariant generalized translation-invariant valuations is spanned by certain explicit $\mu_k^{\pm} \in \mathrm{Val}_k^{-\infty}(\mathbb{R}^{p,q})$, $1 \leq k \leq n-1$, together with χ and vol.

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They can be naturally restricted to subspaces. For $\dim E = k$, $Q|_E$ of signature (a, b), $a + b = k$:

$$\mu_k^+|_E = \begin{cases} \mathrm{vol}_E, & b \equiv 0 \mod 4 \\ -\mathrm{vol}_E, & b \equiv 2 \mod 4 \\ 0, & b \equiv 1 \mod 2 \end{cases}$$

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Pseudo-Riemannian Lipschitz-Killing curvatures

Theorem (Bernig-F.-Solanes), in progress

- Let M^{n+1} have a pseudo-metric of signature $(p+1, q)$, both positive. Then one can define generalized curvature measures $LK_k^{\pm} \in \mathcal{C}_k^{-\infty}(M)$ $(0 \leq k \leq n)$, $LK_{n+1} = \mathrm{vol}$, canonically associated to the metric. On $\mathbb{R}^{p+1,q}$, they globalize to the previously defined $\chi, 0, (\mu_k^{\pm})_{k=1}^n$, vol.

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2. Whether a Fu-Wannerer - type characterization of the $LK_k^{\pm}$ holds remains unknown.

Applications

Theorem: Full version of pseudo-Riemannian Chern-Gauss-Bonnet

For $M^{p,q}$, $n = p + q$, there is an explicit generalized form $\omega \in \Omega^{n-1}_{-\infty}(\mathbb{P}_+(TM))$ such that for nice $X \subset M$, $\chi(X) = (2\pi)^{-n/2} \int_X Pf(\tilde{\Omega}) + \int_{NX} \omega$. Here again $\tilde{\Omega}_{ij} = \Omega_{ij}\epsilon_j$.

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Theorem: Valuations on the pseudo-sphere

The invariant generalized valuations $\mathcal{V}^{-\infty}(S_{\pm}^{p,q})^{\mathrm{O}(p+1,q)}$ are spanned by the L-K valuations.

Contact manifolds

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Example

- $S^{2n+1} \subset \mathbb{C}^{n+1}$, with $T_x S^{2n+1} = x^\perp$ and $H_x S^{2n+1} = (\mathbb{C}x)^\perp \subset T_x S^{2n+1}$.
- There is no linear contact space (a translation-invariant horizontal distribution is integrable). The "standard" contact structure on $\mathbb{R}^{2n+1}$ is a stereographic projection of the contact sphere S^{2n+1}.

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If $N \subset (M, H)$ intersects the horizontal distribution transversally, it has a naturally induced DH structure.

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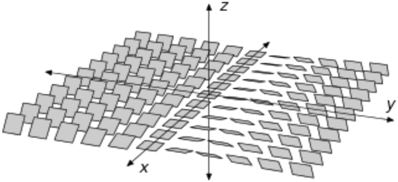
Example - the standard $\mathbb{R}^3$

Consider the standard $(\mathbb{R}^3, dz + ydx = 0)$.

Figure: $\alpha = dz + xdy$



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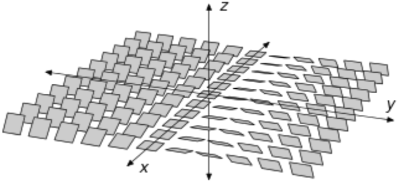

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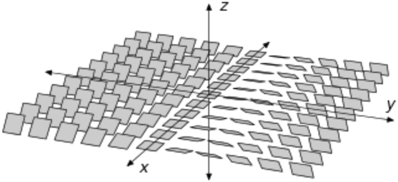

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- For a sphere of radius R, $\phi_2(S_R) = 2R^2$.

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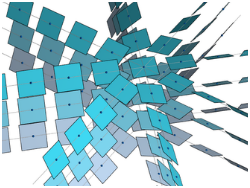

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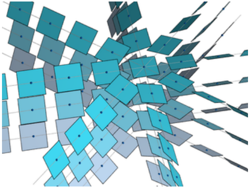

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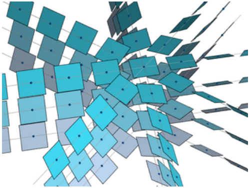
$$\mathrm{CA}_2(F, 0) = 4(1 + \det H^2 f(0))^{-1} = 4(1 + \kappa_F(0))^{-1}$$

Example - the other standard $\mathbb{R}^3$

Let $\mathbb{R}^3$ be equipped with the standard contact structure given by the form $\alpha = dz + xdy - ydx$. It is the stereographic projection of the contact $S^3 \subset \mathbb{C}^2$.

Figure: $\alpha = dz + xdy - ydx$



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- For a sphere of radius R, $\phi_2(S_R) = 8(1 + \frac{1}{R^2})^{-1}$.

Canonic valuations on contact/DH manifolds

Theorem (F. '17+) - Existence and uniqueness of contact Lipschitz-Killing valuations

Any contact manifold M^{2n+1} admits a canonic family of independent generalized valuations: $\phi_{2k} \in \mathcal{W}_{2k}^{-\infty}(M)$, $k = 0, \dots, n$.
Together, (ϕ_{2k}) span the full subspace of generalized valuations invariant under $Cont(M)$.

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$\mathrm{Val}^{-\infty}(\mathfrak{h}_{2n+1}^*)^{Aut(\mathfrak{h}_{2n+1})} = \mathrm{Span}\{\phi_{2k}\}_{k=0}^n$, $\phi_{2k} \in \mathrm{Val}_{2k}^{-\infty}(\mathfrak{h}_{2n+1}^*)$.

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Unlike the (pseudo)metric setting, the ϕ_k don't seem to be the result of the globalization of naturally defined curvature measures.
Also unlike the pseudo-Riemannian setting, not all L-K valuations appear in $\mathfrak{h}_{2n+1}^*$, providing an obvious obstruction to a DH Nash theorem.

The End

Thanks for listening!