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Valuations on spaces of functions

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Preamble

The subject of this talk is relatively new, at least in the community of convex geometry. I decided to take the opportunity to present it at this school, hoping to enlarge the group of people working on it.

The subject is mainly analytic, but it is closely related to convex geometry by several points of view, as I will try to show.

Valuations on spaces of functions

Definition. *Let* $\mathbf{X}$ *be a space of functions. A mapping* $\mathrm{Z}\colon \mathbf{X} \to \mathbb{R}$ *is called a* **valuation**, *if*

$$\mathrm{Z}(f \vee g) + \mathrm{Z}(f \wedge g) = \mathrm{Z}(f) + \mathrm{Z}(g),$$

for every $f, g \in X$, *such that* $f \vee g, f \wedge g \in \mathbf{X}$.

Here

$$f \vee g = \max\{f, g\}, \quad f \wedge g = \min\{f, g\}.$$

Notice the analogy with the definition of valuation on a family of sets $\mathbf{S}$. A mapping $\mathrm{Y}\colon \mathbf{S} \to \mathbb{R}$ is a valuation if:

$$\mathrm{Y}(A \cup B) + \mathrm{Y}(A \cap B) = \mathrm{Y}(A) + \mathrm{Y}(B),$$

for every $A, B \in \mathbf{S}$ such that $A \cup B, A \cap B \in \mathbf{S}$.

An example

Let $\mathbf{X} = C(\mathbb{S}^{n-1})$. The mapping $\mathrm{Z}\colon \mathbf{X} \to \mathbb{R}$ defined by

$$\mathrm{Z}(f) = \int_{\mathbb{S}^{n-1}} f(x)\,\mathrm{d}x$$

is a valuation (the integration is made w.r.t. the $(n-1)$ - dimensional Hausdorff measure on $\mathbb{S}^{n-1}$).

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Proof:

$$A = \{x\colon f(x) > g(x)\}, \quad B = \{x\colon f(x) \le g(x)\}.$$

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} (f \vee g)(x)\,\mathrm{d}x &= \int_A f(x)\,\mathrm{d}x + \int_B g(x)\,\mathrm{d}x \\
\int_{\mathbb{S}^{n-1}} (f \wedge g)(x)\,\mathrm{d}x &= \int_A g(x)\,\mathrm{d}x + \int_B f(x)\,\mathrm{d}x.
\end{aligned}$$

□

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The same is true for the mappings

$$f \mapsto \int_{\mathbb{S}^{n-1}} f^2(x)\,\mathrm{d}x, \quad \int_{\mathbb{S}^{n-1}} e^{f(x)}\,\mathrm{d}x, \dots$$

In general, if $K\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, then

$$\mathrm{Z}(f) = \int_{\mathbb{S}^{n-1}} K(f(x))\,\mathrm{d}x$$

is a valuation on $C(\mathbb{S}^{n-1})$.

Remark. With rare exceptions, all known examples of valuations on spaces of functions can be represented as integrals.

More on this example

The valuation $\mathrm{Z}\colon C(\mathbb{S}^{n-1}) \to \mathbb{R}$

$$\mathrm{Z}(f) = \int_{\mathbb{S}^{n-1}} K(f(x))\,\mathrm{d}x \qquad (\star)$$

is:

- **continuous** with respect to natural topology on $C(\mathbb{S}^{n-1})$ (uniform convergence);
- **rotation invariant**: $\mathrm{Z}(f) = \mathrm{Z}(f \circ \rho)$, for every $\rho \in \mathrm{O}(n)$.

Theorem. [Villanueva '16; Tradacete and Villanueva '17]. *Every continuous and rotation invariant valuation* Z *on* $C(\mathbb{S}^{n-1})$ *can be written in the form* $(\star)$ *for some continuous function* K.

General setting

The spaces under considerations are formed by functions defined either on $\mathbb{R}^n$ or on $\mathbb{S}^{n-1}$.

Examples: $L^p(\mathbb{R}^n)$, $\{$convex functions on $\mathbb{R}^n\}$, $C(\mathbb{S}^{n-1})$ as in the previous example...

And the valuations Z are usually **continuous** and have some geometric invariance:

- *rotation invariance*:

$$\mathrm{Z}(u \circ \rho) = \mathrm{Z}(u), \quad \forall\, \rho \in \mathrm{O}(n),$$

- and/or *translation invariance*:

$$\mathrm{Z}(u(\cdot)) = \mathrm{Z}(u(\cdot - x_0)) \quad \forall\, x_0 \in \mathbb{R}^n.$$

Aim (typically): to classify all possible valuations with these features.

Motivations

- Mainly curiosity, I would say.

- Valuations may be seen as **measures of the size of a function**. This could be helpful for spaces of functions which do not have a natural notion of norm. An example in this sense is the space of convex functions.

- Another motivation is to explore the **affinities with the theory of valuations on convex bodies**, for spaces of functions related to convexity; e.g. convex functions, log-concave functions, quasi-concave functions.

Contributions

- Ludwig '11: *Valuations on function spaces*.
- Tsang '10 (and related previous works – see later); Li and Ma '17: L^p-spaces.
- Baryshnikov, '11; Baryshnikov, Ghrist and Wright, '13: *definable functions*.
- Ludwig '12; Ma '16: Sobolev spaces.
- Wang '14: spaces of functions with bounded variation.
- Kone '14: Orlicz spaces.
- Tradacete '15; Tradacete and Villanueva '17: $C(\mathbb{S}^{n-1})$, and more general spaces (Banach lattices).
- Cavallina and C. '15, Ludwig, Mussnig and C. '17, '18; Alesker, '17: convex functions.
- Lombardi and C. '17; Lombardi, Parapatits and C. '18: quasi-concave functions.

Outline

The rest of this talk is a description of the results that have been achieved for valuations defined on:

- Lebesgue spaces (also in connection with valuations on star-shaped sets);
- continuous functions on $\mathbb{S}^{n-1}$ (and beyond);
- quasi-concave functions;
- convex functions.

Valuations on star-shaped sets

Klain ('96, '97) initiated the study of valuations on *star-shaped sets*, as a counterpart of valuations on convex bodies.

Given a star-shaped set $S \subset \mathbb{R}^n$, let ρ_S denote its radial function.

Klain worked in the class $\mathbf{S}^n$ of star-shaped sets having radial function in $L^n(\mathbb{S}^{n-1})$, and proved the following result.

Theorem. $\tilde{\mathrm{Z}}\colon \mathbf{S}^n \to \mathbb{R}$ *is a continuous and rotation invariant valuation, if and only if it can be represented in the form*

$$\mathrm{Z}(S) = \int_{\mathbb{S}^{n-1}} K(\rho_S(x))dx, \quad \forall\, S \in \mathbf{S}^n,$$

where $K\colon [0,\infty)$ *is continuous and verifies a suitable growth condition at infinity.*

Froms sets to functions

Let S_1 and S_2 be star-shaped sets. Then

$$\rho(S_1 \cup S_2, \cdot) = \rho(S_1, \cdot) \vee \rho(S_2, \cdot),$$

$$\rho(S_1 \cap S_2, \cdot) = \rho(S_1, \cdot) \wedge \rho(S_2, \cdot).$$

Therefore if $\tilde{\mathrm{Z}}$ is a valuation on star-shaped sets and $\mathbf{X}$ is the class of radial functions, the mapping $\mathrm{Z} \colon \mathbf{X} \to \mathbb{R}$:

$$\mathrm{Z}(\rho) = \tilde{\mathrm{Z}}(S), \quad \text{with } \rho = \rho_S$$

is a valuation on $\mathbf{X}$. Hence Klain's results reads as:

Theorem. $\mathrm{Z} \colon L^n_+(\mathbb{S}^{n-1}) \to \mathbb{R}$ *is a continuous and rotation invariant valuation if and only if it can be represented in the form*

$$\mathrm{Z}(f) = \int_{\mathbb{S}^{n-1}} K(f)dx, \quad \forall f \in L^n_+(\mathbb{S}^{n-1})$$

where $K \colon [0, \infty)$ *is continuous and verifies a suitable growth condition at infinity.*

The L^p case

The previous result was extended by Tsang ('10), to the space $L^p(\mathbb{S}^{n-1})$, $1 \leq p < \infty$. In the same paper a corresponding theorem is proved for $L^p(\mathbb{R}^n)$.

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Theorem. $\mathrm{Z}\colon L^p(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and translation invariant valuation, if and only if it can be represented in the form*

$$\mathrm{Z}(f) = \int_{\mathbb{R}^n} K(f(x))dx$$

where $K \in C(\mathbb{R})$ *verifies a suitable growth condition at infinity.*

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where $K \in C(\mathbb{R})$ *verifies a suitable growth condition at infinity.*

Remark. The results by Klain and Tsang can be retrieved in previous works concerning *orthogonally additive mappings*, in the realm of functional analysis. Moreover, the recent work by Tradacete and Villanueva on valuations on Banach lattices (Banach spaces endowed with an appropriate partial order relation), encompass Tsang's classification results.

The case of $C(\mathbb{S}^{n-1})$

Villanueva ('16) and Tradacete and Villanueva ('17) extended the theorem of Klain to valuations on star-shaped sets with **continuous** radial functions.

The functional counterpart of one of their result reads as follows:

Theorem. $\mathrm{Z}\colon C(\mathbb{S}^{n-1}) \to \mathbb{R}$ *is a continuous and rotation invariant valuation if and only if there exists a function* K *continuous on* $\mathbb{R}$ *such that*

$$\mathrm{Z}(f) = \int_{\mathbb{S}^{n-1}} K(f(x))dx \quad \forall f \in C(\mathbb{S}^{n-1}).$$

The previous result invites to analyse spaces of functions defined on $\mathbb{S}^{n-1}$, having more regularity than continuity.

Beyond $C(\mathbb{S}^{n-1})$

Let

$$\mathrm{Lip}(\mathbb{S}^{n-1}) = \{f \colon \mathbb{S}^{n-1} \to \mathbb{R} \mid f \text{ is Lipschitz}\}.$$

In collaboration with Pagnini, Tradacete and Villanueva, we are studying valuations on $\mathrm{Lip}(\mathbb{S}^{n-1})$, which are continuous with respect to a suitable topology, and rotation invariant.

We expect a classification based on integral functionals depending on the involved function **and on its gradient**:

$$\mathrm{Z}(f) = \int_{\mathbb{S}^{n-1}} K(f(x), \|\nabla f(x)\|) \, \mathrm{d}x.$$

We arrived to some partial results, for which I refer to the poster of Daniele Pagnini.

Quasi-concave functions

A non-negative function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called **quasi-concave** if for every $t > 0$:

$$L_t(f) := \{x\colon f(x) \geq t\}$$

is compact and convex (i.e. a convex body), or it is empty.

Familiar examples are:

$$f(x) = e^{-\|x\|^2}; \quad f(x) = \text{characteristic function of a convex body}$$

(more generally, all **log-concave functions** are quasi-concave).

The space $\mathrm{QC}(\mathbb{R}^n)$ of quasi-concave function can be endowed with a natural topology (induced by hypo-convergence).

Valuations on quasi-concave functions

Various structure results can be proved for valuations on $\mathrm{QC}(\mathbb{R}^n)$.

- A Hadwiger type characterization for continuous and **rigid motion invariant** valuations [Lombardi and C., 2017].
- A homogeneous McMullen type decomposition for continuous and **translation invariant** valuations [Lombardi, Parapatits and C., 2018].

I will give more details about the first result.

“Intrinsic volumes” of quasi-concave functions

Fix and index $i \in \{0, 1, \dots, n\}$. Let $f \in \mathrm{QC}(\mathbb{R}^n)$; for every $t > 0$ we may consider

$$t \mapsto V_i(L_t(f)) = V_i(\{x \colon f(x) \geq t\}),$$

where V_i is the i-th intrinsic volume.

The mapping $\mathrm{Z} \colon \mathrm{QC}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\mathrm{Z}(f) = \int_0^{+\infty} V_i(L_t(f))\phi(t)\,\mathrm{d}t,$$

is a **continuous valuation** provided $\phi \in C((0,\infty))$ and vanishes identically in $(0,\delta)$ for some $\delta > 0$. Z is also **rigid motion invariant**.

Remarks

Let $\mathrm{Z}\colon \mathrm{QC}(\mathbb{R}^n) \to \mathbb{R}$ be defined by

$$\mathrm{Z}(f) = \int_0^{+\infty} V_i(L_t(f))\phi(t)\,\mathrm{d}t.$$

Remark 1. A more general class of functionals was introduced by Milman and Rotem in 2013, as counterpart of mixed volumes for quasi-concave functions.

Remark 2. These simple functionals "almost" saturate the class of continuous and rigid motion invariant valuations on $\mathrm{QC}(\mathbb{R}^n)$, but not exactly.

A refinement of the previous construction

Fix and index $i \in \{0, 1, \dots, n\}$. Let $f \in \mathrm{QC}(\mathbb{R}^n)$;

$$t \mapsto V_i(L_t(f)) = V_i(\{x \colon f(x) \geq t\}),$$

is **decreasing** in t. Its derivative is a (non-positive) measure on $(0, +\infty)$.

Let $\psi \in C(0, +\infty)$ be such that $\psi \equiv 0$ in $(0, \delta)$ for some $\delta > 0$. The mapping $\mathrm{Z} \colon \mathrm{QC}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\mathrm{Z}(f) = \int_0^{+\infty} \psi(t) \, \mathrm{d} V_i(L_t(f)), \qquad (\star)$$

is a continuous and rigid motion invariant valuation on $\mathrm{QC}(\mathbb{R}^n)$.

Remark. When ψ is smooth enough, $(\star)$ can be written in the form of the previous slide by an integration by parts.

A Hadwiger type theorem for valuation on $\mathrm{QC}(\mathbb{R}^n)$

Theorem. *Let* $\mathrm{Z}\colon QC(\mathbb{R}^n) \to \mathbb{R}$ *be a continuous and rigid motion invariant valuation. Then* Z *can be represented in the form*

$$\mathrm{Z}(f) = \sum_{i=0}^{n} \int_0^{+\infty} \psi_i(t)\, \mathrm{d}V_i(\{x\colon f(x) \geq t\}), \quad \forall f \in \mathrm{QC}(\mathbb{R}^n),$$

where $\psi_0, \ldots, \psi_n \in C(0, \infty)$ *and, except for* ψ_0*, they vanish identically in* $(0, \delta)$ *for some* $\delta > 0$*.*

Remark 1. The proof relies strongly on Hadwiger's theorem for convex bodies.

Remark 2. The advantage here is that the class $\mathrm{QC}(\mathbb{R}^n)$ is ample (in general, the larger is the class of functions, the easier is to classify all possible valuations). In particular it contains characteristic functions of convex bodies and their positive multiples.

Valuations on convex functions

The study of valuations on convex functions started very recently. It received contributions from: Alesker, Cavallina, Ludwig, Mussnig, C., in various directions.

I will describe some results concerning (real-valued) valuations, obtained in collaboration with Ludwig and Mussnig, as well as possible developments of the theory.

Convex functions

There are various spaces of convex functions that one may consider. One is $\mathrm{Conv}(\mathbb{R}^n)$, which is formed by functions $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ having the following properties:

1. f is convex;
2. f is lower semi-continuous;
3. f is *coercive*:

$$\lim_{\|x\|\to+\infty} f(x) = +\infty.$$

Other classes are:

- finite convex functions on $\mathbb{R}^n$;
- *super-coercive* convex functions on $\mathbb{R}^n$;
- convex functions on $\mathbb{R}^n$ (possibly infinite) without growth condition at infinity.

Each of these spaces can be equipped with the topology induced by the epi-convergence.

Example of valuations on $\mathrm{Conv}(\mathbb{R}^n)$

Example 1. $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\mathrm{Z}(f) = \min_{\mathbb{R}^n} f$$

is a continuous, translation and $\mathrm{SL}(n)$ invariant valuation. More generally, the same holds for

$$\mathrm{Z}(f) = K(\min_{\mathbb{R}^n} f)$$

if $K\colon \mathbb{R} \to \mathbb{R}$ is continuous.

Example 2. $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\mathrm{Z}(f) = \int_{\mathrm{dom}(f)} K(f(x))\,\mathrm{d}x$$

is a continuous, translation and $\mathrm{SL}(n)$ invariant valuation provided $K\colon \mathbb{R} \to \mathbb{R}$ is continuous and the following integral is **finite**:

$$\int_0^{+\infty} t^{n-1} K(t)\,\mathrm{d}t.$$

A first result

Theorem (Ludwig, Mussnig, C., '17). $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, translation and* $SL(n)$*-invariant valuation if and only if there exists* $K_0, K_1 \in C(\mathbb{R})$ *such that*

$$\mathrm{Z}(f) = K_0(\min_{\mathbb{R}^n} f) + \int_{\mathrm{dom}(f)} K_1(f(x))\,\mathrm{d}x \quad \forall\, f \in \mathrm{Conv}(\mathbb{R}^n)$$

and

$$\int_0^{+\infty} t^{n-1} K(t)\, dt$$

is finite.

This results reproduces in a functional setting one of the classification results of $\mathrm{SL}(n)$ invariant valuations on convex bodies due to Ludwig. Ludwig's results are also crucial in the proof.

New examples of valuations on convex functions

In 2017 two new classes of valuations on convex functions have been introduced, the first by Alesker and the second by Ludwig, Mussnig and C. (Hessian valuations).

They act on different classes of convex functions and have different invariance properties, but both involve **second derivatives** of convex functions, either in the classical or in the generalised sense (i.e. as measures).

Alesker's valuations

They are of the form (for $f \in C^2(\mathbb{R}^n)$ and convex)

$$Z_A(f) = \int_{\mathbb{R}^n} b(x)\, \mathrm{D}(M_1(x), \dots, M_{n-i}(x), D^2 f(x)[i])\, \mathrm{d}x,$$

where b is a given function, D denotes the **mixed discriminant**, $i \in \{0, \dots, n\}$, $M_1, \dots, M_{n-i}$ are symmetric matrix-valued functions on $\mathbb{R}^n$.

Alesker proved that under suitable assumptions on b and the M_j's, Z_A can be extended to a continuous valuation on the class of **finite** convex functions.

Z_A is invariant under the addition of affine functions:

$$Z_A(f) = Z_A(f+g) \quad \forall\, g \text{ affine},$$

and i-homogeneous: $Z_A(\lambda f) = \lambda^i\, Z_A(f)$.

Hessian valuations

They are of the form (for $f \in C^2(\mathbb{R}^n)$ and convex)

$$\mathrm{Z}_H(f) = \int_{\mathbb{R}^n} K(x, f(x), \nabla f(x))\, [D^2 f(x)]_i \, \mathrm{d}x,$$

where K is a continuous function, $i \in \{0, \dots, n\}$ and $[D^2 f(x)]_i$ is the i-th elementary symmetric function of the eigenvalues of $D^2 f(x)$.

Under suitable assumption on K, Z_H can be extended to a continuous valuations on $\mathrm{Conv}(\mathbb{R}^n)$.

Moreover, depending on the choice of K, it may be translation and/or rotation invariant and/or invariant under the addition of constants.

Currently, our goal is to understand whether results similar to the **Hadwiger characterisation theorem** of rigid motion and continuous valuations on convex bodies, or the **McMullen homogenous decomposition** admit corresponding results in the context of convex functions.

Alesker's and Hessian valuations may play a role in this research, providing counterexamples, or functional counterparts to intrinsic and mixed volumes of convex bodies.

Announcement!

In the summer of 2020 (probably at the beginning of July) there will be a **school** entitled **Convex Geometry**, in Cetraro (south of Italy, on the Tirreno sea). The school is part of the activity of the Fondazione C.I.M.E (Centro Internazionale Matematico Estivo).

You are all warmly invited to participate!