The square negative correlation property on ℓ_p^n balls

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Joint work with Julio Bernués

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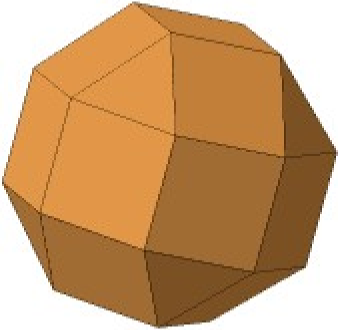
Convex bodies

- $K \subset \mathbb{R}^n$ is called a convex body if it is convex, compact and has non-empty interior.



Isotropic bodies

A convex body $K \subseteq \mathbb{R}^n$ is isotropic if it has volume 1 and

- $\int_K x dx = 0$ (centered at 0)
- $\int_K \langle x, \theta \rangle^2 dx = L_K^2 \quad \forall \theta \in S^{n-1}$.

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Given K we consider a random vector X uniformly distributed in K and, for every $\theta \in S^{n-1}$, the real random variable $\langle X, \theta \rangle$ with density $f_\theta(t) = |K \cap (\theta^\perp + t\theta)|$.

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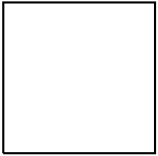

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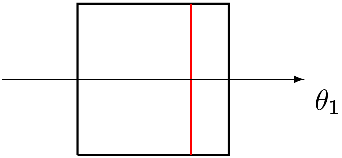

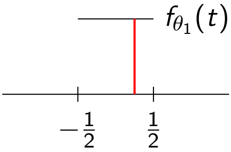

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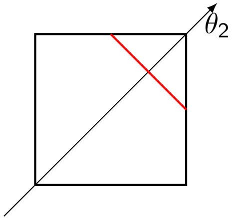

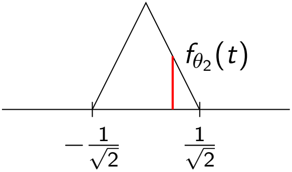

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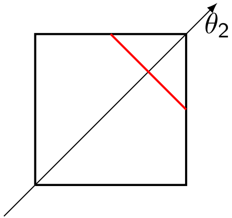

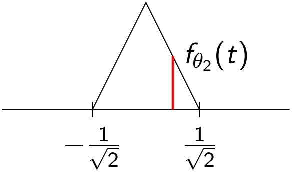

K is isotropic if all the $\langle X, \theta \rangle$ are centered and have the same variance.

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There exists an **absolute** constant C such that for every $K \subseteq \mathbb{R}^n$

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- $L_K \leq C n^{\frac{1}{4}}$ (Klartag 2005, Lee-Vempala 2016)
- True for 1-unconditional bodies, polytopes with number of vertices proportional to the dimension, zonoids, unit balls of finite dimensional Schätten classes...

The central limit problem

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- $G \sim \mathcal{N}(0,1)$

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}} > t \right) - \mathbb{P}(G > t) \right| = 0$$

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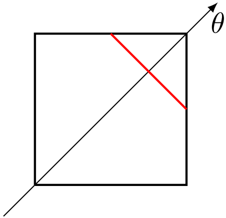

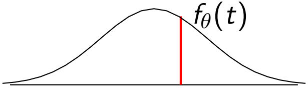

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Let X be a random vector uniformly distributed on $L_K^{-1}K$ with K isotropic. For how many directions $\theta \in S^{n-1}$ can we say

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Sudakov (1978), Diaconis-Freedman (1984), von Weizsäker (1997), Antilla-Ball-Perissinaki (2003).

Shown under a concentration hypothesis:

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Klartag (2007):

$$\mathbb{P}\left(\left| \frac{|X|}{\sqrt{n}} - 1 \right| \geq \frac{c}{n^{\kappa}} \right) \leq Ce^{-n^{\kappa}}$$

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- True for uniformly distributed random vectors on B_p^n (Antilla-Ball-Perissinaki, 2003), Orlicz balls (Wojtaszczyk, 2007), unconditional bodies (Klartag, 2012)

The square negative correlation property

Definition

A centered log-concave random vector $X \in \mathbb{R}^n$ satisfies the square negative correlation property with respect to the orthonormal basis $\{\eta_i\}_{i=1}^n$ if for every $i \neq j$

$$\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2 \leq 0$$

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- X uniformly distributed on $P_{\theta^\perp} B_\infty^n$, any $\{\eta_i\}_{i=1}^n$. A., Bastero (2013).

The square negative correlation property

- X uniformly distributed on K isotropic, $\{\eta_i\}_{i=1}^n$

$$\begin{aligned}\mathrm{Var}|X|^2 &= \sum_{i=1}^n (\mathbb{E}\langle X, \eta_i\rangle^4 - (\mathbb{E}\langle X, \eta_i\rangle^2)^2) \\ &+ \sum_{i \neq j} (\mathbb{E}\langle X, \eta_i\rangle^2 \langle X, \eta_j\rangle^2 - \mathbb{E}\langle X, \eta_i\rangle^2 \mathbb{E}\langle X, \eta_j\rangle^2)\end{aligned}$$

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$$\begin{aligned}\mathrm{Var}|X|^2 &\leq \sum_{i=1}^{n}(\mathbb{E}\langle X,\eta_i\rangle^4 - (\mathbb{E}\langle X,\eta_i\rangle^2)^2) \\ &\leq C\sum_{i=1}^{n}(\mathbb{E}\langle X,\eta_i\rangle^2)^2\end{aligned}$$

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- A random vector uniformly distributed on B_p^n satisfies the SNCP with respect to $\{e_i\}_{i=1}^n$. What about other basis?

Proposition (A., Bernués (2018))

Let X be a random vector uniformly distributed on a 1-symmetric convex body, $\xi_1 = \frac{e_1+e_2}{\sqrt{2}}$, $\xi_2 = \frac{e_1-e_2}{\sqrt{2}}$, and $f : S^{n-1} \times S^{n-1} \to \mathbb{R}$

$$f(\eta_1, \eta_2) = \mathbb{E}\langle X, \eta_1\rangle^2 \langle X, \eta_2\rangle^2 - \mathbb{E}\langle X, \eta_1\rangle^2 \mathbb{E}\langle X, \eta_2\rangle^2.$$

Then, if $\langle \eta_1, \eta_2\rangle = 0$

$$f(\eta_1, \eta_2) = f(e_1, e_2) + 2(f(\xi_1, \xi_2) - f(e_1, e_2)) \sum_{i=1}^n \eta_1(i)^2 \eta_2(i)^2.$$

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- $0 \leq \sum_{i=1}^{n} \eta_1(i)^2 \eta_2(i)^2 \leq \frac{1}{2}$,

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$$F(x) = \frac{\Gamma(5x)\Gamma(x)}{\Gamma(3x)^2.}$$

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$$F(x) = \frac{\Gamma(5x)\Gamma(x)}{\Gamma(3x)^2}.$$
- F is strictly increasing in $(0,1]$ and $F\left(\frac{1}{2}\right) = 3$.

The square negative correlation property on B_p^n

Theorem (A., Bernués (2018))

Let X be a random vector uniformly distributed on B_p^n, $\xi_1 = \frac{e_1+e_2}{\sqrt{2}}$, $\xi_2 = \frac{e_1-e_2}{\sqrt{2}}$. If $\langle \eta_1, \eta_2 \rangle = 0$

$$\begin{array}{lll} f(\xi_1,\xi_2) \le f(\eta_1,\eta_2) \le f(e_1,e_2) & \text{if} & p \ge 2 \\ f(e_1,e_2) \le f(\eta_1,\eta_2) \le f(\xi_1,\xi_2) & \text{if} & p \le 2 \end{array}$$

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The square negative correlation property on B_p^n

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The square negative correlation property on B_p^n

- $$f(\xi_1, \xi_2) = \frac{\Gamma\left(1+\frac{n}{p}\right)\Gamma\left(\frac{3}{p}\right)^2\left(F\left(\frac{1}{p}\right) - 1 - 2\frac{\Gamma\left(1+\frac{n}{p}\right)\Gamma\left(1+\frac{n+4}{p}\right)}{\Gamma\left(1+\frac{n+2}{p}\right)^2}\right)}{2\Gamma\left(1+\frac{n+4}{p}\right)\Gamma\left(\frac{1}{p}\right)^2}$$

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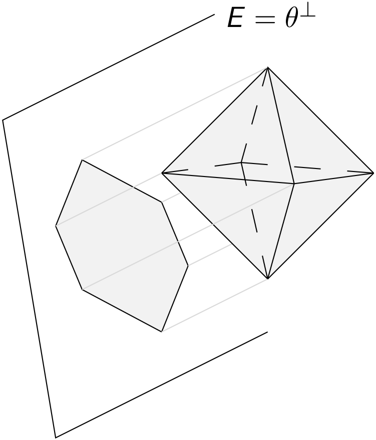
- If $1 \leq p < 2$ there exists n_0 such that if $n \geq n_0$ $f(\xi_1,\xi_2) > 0$.

Cauchy's formula

$P_{\theta^\perp} K$



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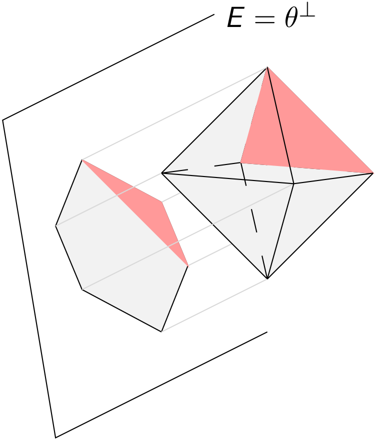

$$\int_{P_E K} f(x)dx =$$

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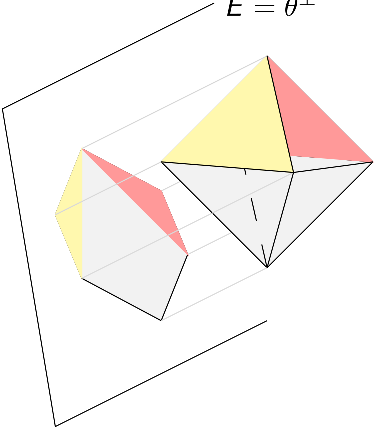

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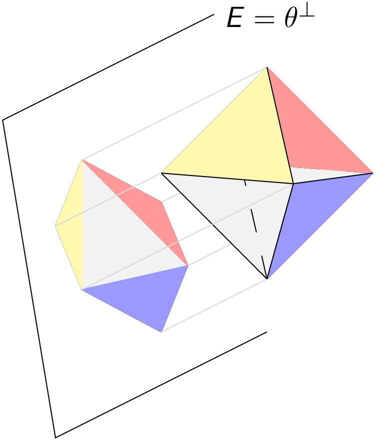

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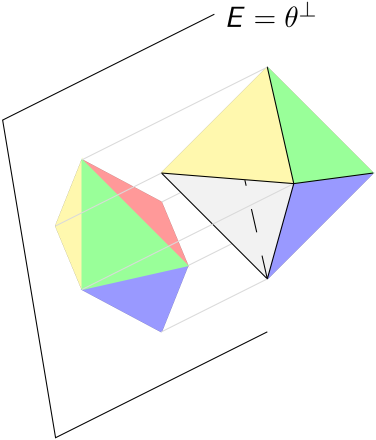
$$P_{\theta^\perp} K = \bigcup_{F \in \mathcal{F}_{n-1}(K) : \langle \nu(F), \theta \rangle \geq 0} P_E F$$



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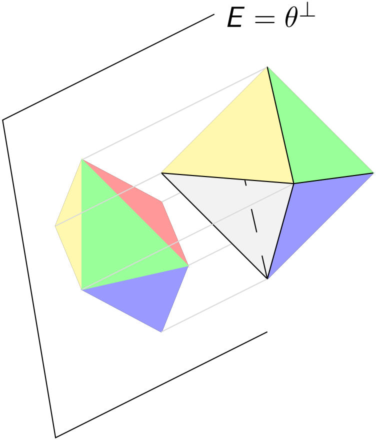

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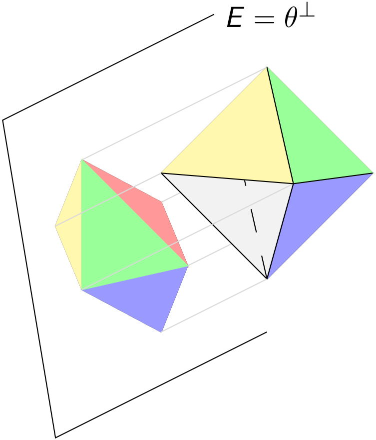

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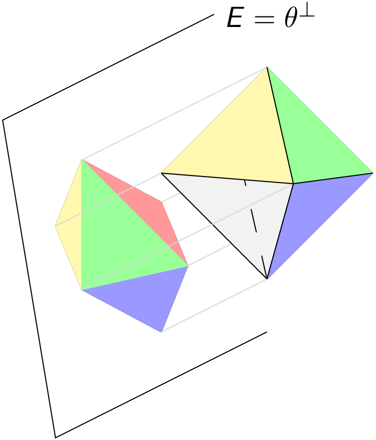

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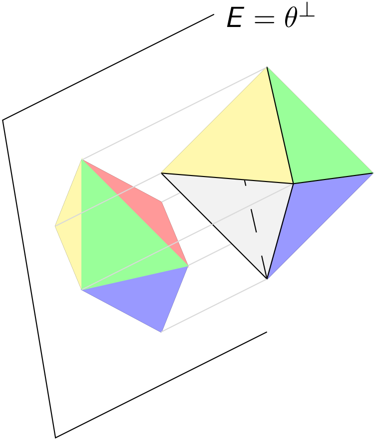

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$$= \frac{1}{2} \int_{\partial K} f(P_E y) |\langle \nu(y), \theta \rangle| d\sigma_K(y)$$

$d\sigma_K$ Hausdorff measure on ∂K

Random vectors on hyperplane projections of B_p^n

- By Cauchy's formula, if X is uniformly distributed on $P_{\theta^\perp}(B_p^n)$

$$\mathbb{E}f(X) = \frac{\int_{\partial B_p^n} f(P_{\theta^\perp}(x)) \frac{|\langle \nabla \|\cdot\|_p(x), \theta\rangle|}{|\nabla \|\cdot\|_p(x)|} d\sigma_p^n(x)}{\int_{\partial B_p^n} \frac{|\langle \nabla \|\cdot\|_p(x), \theta\rangle|}{|\nabla \|\cdot\|_p(x)|} d\sigma_p^n(x)}.$$

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- Since $d\sigma_p^n(x) = n|B_p^n||\nabla\| \cdot \|_p(x)| d\mu_p^n(x)$ (Naor, Romik, 2003)

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Probabilistic representation of $d\mu_p^n$ (Schechtman-Zinn (1990)

Let $g_1, \ldots, g_n$ be independent copies of a random variable with density

$$\frac{e^{-|t|^p}}{2\Gamma\left(1+\frac{1}{p}\right)}$$

and

$$S = \left(\sum_{i=1}^{n} |g_i|^p\right)^{\frac{1}{p}}.$$

Then

- The random vector $\frac{g}{S} = \left(\frac{g_1}{S}, \ldots, \frac{g_n}{S}\right)$ and the random variable S are independent.
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If X is uniformly distributed on $P_{\theta^\perp}(B_p^n)$

$$\mathbb{E}f(X) = \frac{\mathbb{E}f\left(P_H\left(\frac{g}{S}\right)\right)\phi}{\mathbb{E}\phi}, \qquad \phi = \left|\sum_{i=1}^{n} \frac{|g_i|^{p-1}}{S^{p-1}} \mathrm{sgn}(g_i)\theta_i\right|$$

The square negative correlation property on hyperplane projections of B_p^n

$\theta_0 = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$, $P_{\theta_0^\perp}(B_p^n)$ is isotropic.

Theorem (A., Bernués, 2018)

Let X be a random vector uniformly distributed on $P_{\theta_0^\perp}(B_p^n)$,
$\xi_1 = \frac{e_1 - e_2 + e_3 - e_4}{2}$, $\xi_2 = \frac{e_1 - e_2 - e_3 + e_4}{2}$, $\overline{\xi}_1 = \frac{e_1 - e_2}{\sqrt{2}}$, $\overline{\xi}_2 = \frac{e_3 - e_4}{\sqrt{2}}$ and
$f : S_{\theta_0^\perp} \times S_{\theta_0^\perp} \to \mathbb{R}$

$$f(\eta_1, \eta_2) = \mathbb{E}\langle X, \eta_1\rangle^2 \langle X, \eta_2\rangle^2 - \mathbb{E}\langle X, \eta_1\rangle^2 \mathbb{E}\langle X, \eta_2\rangle^2.$$

If $p \geq 2$ there exists $n_0(p)$ such that if $n \geq n_0$ and $\langle \eta_1, \eta_2\rangle = 0$

$$f(\xi_1, \xi_2) \leq f(\eta_1, \eta_2) \leq f(\overline{\xi}_1, \overline{\xi}_2).$$

If $1 \leq p \leq 2$ there exists $n_1(p)$ such that if $n \geq n_1$ and $\langle \eta_1, \eta_2\rangle = 0$

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The square negative correlation property on hyperplane projections of B_p^n

Corollary (A., Bernués, 2018)

Let X be a random vector uniformly distributed on $P_{\theta_0^\perp}(B_p^n)$. There exists $n_0(p)$ such that for all $n \geq n_0$ X satisfies the SNCP with respect to every orthonormal basis in $\theta_0^\perp$.