Abstract

In this work we propose to improve Brunn-Minkowski and Rogers-Shephard inequalities in terms of the asymmetry measure of Minkowski. We do a first step by computing some bounds via stability results of those inequalities.

Definitions and properties

Let $K^n$ be the set of full-dimensional compact and convex sets in $\mathbb{R}^n$.

- A simplex $\Delta$ in $\mathbb{R}^n$ is the convex hull of $n + 1$ affinely independent points.
- Let the Minkowski sum of $K$ and $L$ be defined by
  \[ K + L := \{ x + y \in \mathbb{R}^n \mid x \in K, y \in L \}. \]

Let $vol(K)$ be the $n$-dimensional volume (or Lebesgue measure) of $K$.

Let the Minkowski measure of asymmetry of $K$ be defined by

\[ s(K) := \inf \{ \lambda \geq 1 \mid -K \subset x + \lambda \cdot K, \text{ for some } x \in \mathbb{R}^n \}. \]

Figure 2: $K \subset s(K)(-K)$ and $\Delta \subset 2(-\Delta)$ for a triangle $\Delta$

Lemma: Let $K \in \mathcal{K}^n$. Then $1 \leq s(K) \leq n$. Moreover, $s(K) = 1$ iff $K = x - K$, $x \in \mathbb{R}^n$ and $s(K) = n$ iff $K$ is a simplex.

Volume and Minkowski addition

- The Brunn-Minkowski inequality (BM) (cf. [4,5]) states for $K, L \in \mathcal{K}^n$ that
  \[ vol(K + L)^\frac{1}{n} \geq vol(K)^\frac{1}{n} + vol(L)^\frac{1}{n}, \]
  Moreover, equality holds iff $L = x + \lambda \cdot K$, for some $x \in \mathbb{R}^n$ and $\lambda > 0$.
- The Rogers-Shephard inequality (RS) (cf. [6]) states for $K, L \in \mathcal{K}^n$ that
  \[ vol(K + L)vol(K \cap (-L)) \leq \left( \frac{2n}{n} \right)^n vol(K)vol(L), \]
  Moreover, equality holds iff $L = -K$ is a simplex (cf. [3]).
- Letting $L = -K$, then (BM) and (RS) summarizes as
  \[ 2^n \leq \frac{vol(K - K)}{vol(K)} \leq \frac{2n}{n}. \]
  Moreover, on LHS iff $K = x - K, x \in \mathcal{K}^n$, resp. on RHS iff $K$ is a simplex.

**QUESTION:** Let $K \in \mathcal{K}^n$ and $s \in [1, n]$ s.t. $s = s(K)$.

What are the smallest $C(s) > 0$ and largest $c(s) > 0$ s.t.

\[ c(s) \leq \frac{vol(K - K)}{vol(K)} \leq C(s)? \]

References


Stability of Brunn-Minkowski and Rogers-Shephard

- A stability version of Brunn-Minkowski inequality (cf. [8,9]) states for $K \in \mathcal{K}^n$ that
  \[ \frac{vol(K - K)}{vol(K)} \leq 2^n \left( 1 + \frac{A(K)^2}{14n^24^{n-1}} \right)^n, \]
  where $A(K) = \inf_{x \in \mathbb{R}^n} \frac{vol(K - x)}{vol(K)}$.
- A stability version of Rogers-Shephard inequality (cf. [7]) states for $K \in \mathcal{K}^n$ that
  \[ 1 - n(d_{BM}(K, \Delta)) \leq \left( \frac{2n}{n} \right)^{-1} \frac{vol(K - K)}{vol(K)} \leq 1 - \frac{d_{BM}(K, \Delta) - 1}{n^{n-1}}, \]
  where the Banach-Mazur distance between $K$ and a simplex $\Delta$ is defined by
  \[ d_{BM}(K, \Delta) = \inf_{x \in \mathbb{R}^n \setminus M \in \mathcal{K}^n} \{ \lambda \geq 1 \mid \Delta \subset x + M(K) \subset y + \lambda \Delta \}. \]

References


First answers the question

**Theorem 1:** Let $K \in \mathcal{K}^n$ and let $s = s(K)$. Then

\[ c(s) \geq \begin{cases} \frac{2^n}{n} \left( 1 - \frac{1}{n} \left( 1 - \frac{n-s}{n} \right)^{n-1} \right)^n & \text{if } 1 < s < n, \\ \frac{(2n)^n}{(1 - 4n^2(n-s))} & \text{if } n - \frac{1}{n} < s < n, \end{cases} \]

and

\[ C(s) \leq \begin{cases} (1 + s)^n & \text{if } 1 < s < n, \\ \left( \frac{2n}{n} \right)^{-1} \left( 1 - \frac{n-s}{n} \right)^{n-1} & \text{if } n - \frac{1}{n} < s < n. \end{cases} \]

**Remark:** The 1. (resp. 2.) upper and lower bounds are specially good when $s(K) \approx 1$ (resp. $s(K) \approx n$).

The **diagram** $f : [1, n] \to \left[ 2^n, \left( \frac{2n}{n} \right)^n \right]$ is defined by $f(K) := \left( s(K), \frac{vol(K - K)}{vol(K)} \right)$.

**Theorem 2:** $f(K^n)$ is simply connected and contains (1, 2$n$) and $(n, (\frac{2n}{n})^n)$.

References


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Stability results in the planar case

Let $\Delta$ be a regular simplex with center $0$.

- $K_{\Delta} := \Delta \cap (s(-\Delta))$.
- $\Delta_{\Delta} := \cap (s(-\Delta))$.

- $f(K_{\Delta})$ contains the dark grey area, is contained in the light grey one.
- The lower boundary of the dark grey area is given by $f(K_{\Delta}) = \{ s, \frac{2(s+1)^2}{2s} \}$. The upper boundary of the dark grey area is given by $f(C_{\Delta}) = \{ s, 2(s+1) \}$. The blue and red dashed lines are given by Theorem 1.

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