GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Cone valuations, Gram's relation, and Flag-angles

Spencer Backman, Sebastian Manecke, Raman Sanyal

We study linear relations of interior and exterior angle sums of polytopes with respect to simple cone valuations. For all these cone angles we prove that only one linear relation exists, for the interior angle this being the Gram-relation. The uniqueness follows from a connection between angle sums and the combinatorics of zonotopes. Surprisingly for a zonotope, the angle-sums are independent of the notion of angle used and only depend on the combinatorics.
We further introduce flag-angles, an analogue to flag-f-vectors and show that flag-angle sums again exhibit a connection to the combinatorics of zonotopes. This allows us to determine all linear relations on flag-angle vectors.

Solid angles and cone valuations

A generalization of angle in higher dimensions is the notion of solid angle. Given a cone $C \subseteq \mathbb{R}^d$, the solid angle $\alpha(C)$ of this cone is defined as

$$\alpha(C) = \frac{\mathrm{vol}_d\, C \cap B_d(0)}{\mathrm{vol}_d\, B_d(0)}\,,$$

where $B_d(0) \subseteq \mathbb{R}^d$ is the unit ball centered at the origin. Here, we normalized the angle, such that $\alpha(\mathbb{R}^d) = 1$.
To measure angles at a face F of a fulldimensional polytope $P \subseteq \mathbb{R}^d$, we evaluate α on the tangent cone $T_F P = \mathrm{cone}\, P - q$ for some $q \in \mathrm{relint}\, F$:

$$\widehat{\alpha}(F, P) := \alpha(T_F P)\,.$$

For any polytope P, let $\widehat{\alpha}_i(P) = \sum_{\dim F = i} \widehat{\alpha}(F, P)$. The solid angle satisfies the Gram-relation, an analogue of the Euler-relation:

The Gram-relation (Gram 1874, Perles and Shephard 1969)

$$\widehat{\alpha}_0(P) - \widehat{\alpha}_1(P) + \cdots + (-1)^{d-1}\widehat{\alpha}_{d-1}(P) \;=\; (-1)^{d+1}$$

The solid angle is a normalized ($\alpha(\mathbb{R}^d) = 1$) and simple ($\alpha(C) = 0$ for $\dim C < d$) valuation valuation. We call these valuations cone angles. Interior angles can be defined for any cone angle.

Question:

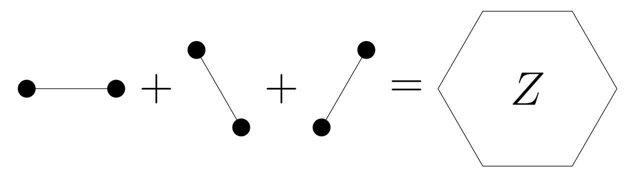
What are the linear relations among the $\widehat{\alpha}_i(P)$'s for α a fixed cone angle?

Theorem: Only the Gram-relation

Let α be a cone angle and P be any d-dimensional polytope, $d > 0$. Then the only linear relations up to scaling satisfied by interior angle sums $\widehat{\alpha}_0(P), \ldots, \widehat{\alpha}_{d-1}(P)$ is the Gram-relation.

Zonotopes

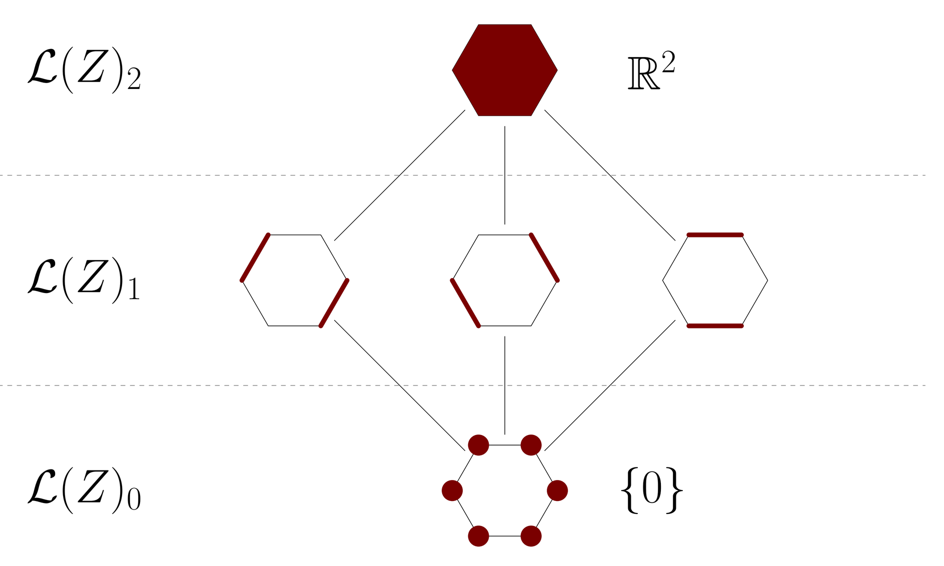
To prove uniqueness the Main Theorem, one has to find a set of polytopes spanning the hyperplane given by the Gram-relations.
A zonotope Z is the Minkowski sum of segments:



To every zonotope Z with face lattice $\mathcal{F}(Z)$ we can associate the lattice of flats $\mathcal{L}(Z)$:

$$\mathcal{L}(Z) := \{\mathrm{aff}\, F - q : F \in \mathcal{F}(Z) \setminus \{\varnothing\}, q \in F\},$$

ordered by inclusion. Let $\mathcal{L}(Z)_i$ be the rank i part of $\mathcal{L}(Z)$.



Angle sums on zonotopes

Theorem: The angle sums of zonotopes are combinatorial

For a d-zonotope Z and any cone angle α, it holds:

$$\widehat{\alpha}_i(Z) = |w_i^{\mathrm{co}}| := \sum_{M \in \mathcal{L}(Z)_i} |\mu(M, \mathbb{R}^d)|$$

where w_i^{co} are the Co-Whitney numbers of the first kind, which are defined using the Möbius function μ of the lattice of flats $\mathcal{L}(Z)$.

In particular the interior angle sums $\widehat{\alpha}_i$ do only depend on the lattice of flats of the zonotope and not on the cone angle α.
Now uniqueness of the Main Theorem follows by providing $d-1$ zonotopes, whose Co-Whitney numbers are spanning.

Flag angles

For a cone angle α, a polytope P, and a subset $S = \{0 \le s_1 < \cdots < s_k \le d-1\}$, define the interior flag-angle

$$\widehat{\alpha}_S(P) = \sum_{F_1 \subsetneq \cdots \subsetneq F_k} \widehat{\alpha}(F_1, F_2) \cdot \ldots \cdot \widehat{\alpha}(F_k, P)$$

where the sum is over all chains of faces of P, such that $\dim F_i = s_i$, $i = 1, \ldots, k$.

Theorem: Relations of the interior flag-angles

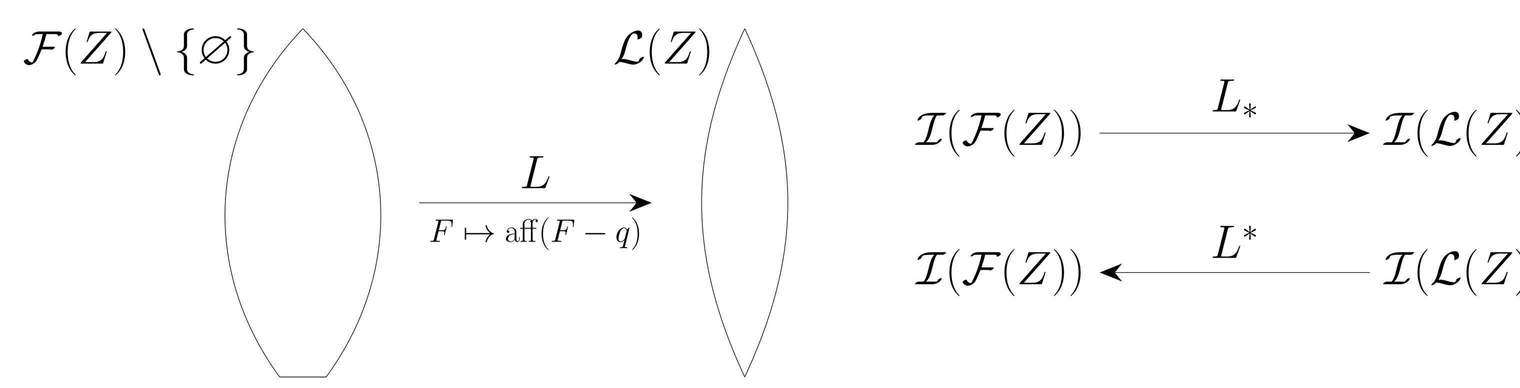
The only linear relations among the flag-angle sums $\widehat{\alpha}_S(P)$, $S \subseteq \{0, 1, \ldots, d-1\}$ are the flag-analogues of the Gram-relation.

Exterior angles

All results also are true for exterior angles, which measure the normal cone

$$\breve{\alpha}(F, P) \;:=\; \alpha(N_F P + N_F P^{\perp})\,.$$

The single relation taking the place of the Gram-relation is the trivial relation $\breve{\alpha}_0(P) = 1$.
This duality is best stated as an algebraic connection between the angles of zonotopes and the ζ- and μ-functions on the lattice of flats. There is an adjoint pair of maps between their incidence algebras:



This correspondence results in a correspond:

Zonotope Z	Lattice of flats $\mathcal{L}(Z)$
Interior angles $\widehat{\alpha}$	Möbius function
Exterior angles $\breve{\alpha}$	Zeta function
Interior angle sums $\widehat{\alpha}_i$	Co-Whitney numbers of the first kind
Exterior angle sums $\breve{\alpha}_i$	Co-Whitney numbers of the second kind
Flag-interior angle sums $\widehat{\alpha}_S$	Co-Flag-Whitney numbers of the first kind
Flag-exterior angle sums $\breve{\alpha}_S$	Co-Flag-Whitney numbers of the second kind

Email: manecke@math.uni-frankfurt.de