Octonion-valued forms: cobblestones on the road to $\mathrm{Val}^{Spin(9)}$

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Normed division algebras

Famously, there are no algebras equipped with a product-compatible norm other then the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

This classical result of Hurwitz is extremely generic. It has to do with e.g. semisimple Lie algebras, certain Jordan algebras, Hopf fibrations or with **compact connected Lie groups acting transitively on a unit sphere** [Borel, Motgomery & Samelson]:

$$SO(n) = SO(\mathbb{R}^n);$$
$$U(n), SU(n) \subset SO(\mathbb{C}^n);$$
$$Sp(n), Sp(n)U(1), Sp(n)Sp(1) \subset SO(\mathbb{H}^n);$$
$$G_2 \subset SO(\operatorname{Im}\mathbb{O});\ Spin(7) \subset SO(\mathbb{O});\ Spin(9) \subset SO(\mathbb{O}^2).$$

Further, this list (almost) agrees with classification of non-symmetric holonomies [Berger, Simons]. Related to the concept of holonomy and also of individual interest are the canonical invariant forms: ω (Kähler), Ω (Kraines), ϕ (associative calibration), Φ (Cayley calibration) and Ψ. They live on $\mathbb{C}^n$, $\mathbb{H}^n$, $\operatorname{Im}\mathbb{O}$, $\mathbb{O}$ and $\mathbb{O}^2$, are of degree 2, 4, 3, 4 and 8 and invariant under $U(n)$, $Sp(n)Sp(1)$, G_2, $Spin(7)$ and $Spin(9)$.

A closer look at $\mathbb{O}$ (and $\mathbb{H}$ too)

The 8-dimensional algebra of **octonions** is just $\mathbb{R}^8$, with the standard basis $\{1, e_1, \dots, e_7\}$, equipped with the involution: $\overline{1} = 1$, $\overline{e_i} = -e_i$, and the algebra product: 1 is the unit, $e_i^2 = -1$, $e_i e_j = -e_j e_i$ for $i \neq j$, and $\left\downarrow \begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{smallmatrix} \right\downarrow$, standing for $e_1 e_2 = e_4$, $e_2 e_4 = e_1$, $e_4 e_1 = e_2$, etc.
The inner product is given by $\langle u, v\rangle = \operatorname{Re}(u\overline{v})$, where $\operatorname{Re}(u) = \frac{1}{2}(u + \overline{u})$. One can check that the induced norm is product-compatible indeed. The subspace of *imaginary* octonions is $\operatorname{Im}\mathbb{O} = \{u \in \mathbb{O}; \operatorname{Re}(u) = 0\}$.

The 4-dimensional algebra of **quaternions** can be viewed as the subalgebra of $\mathbb{O}$ spanned by $\{1, e_1, e_2, e_4\}$. Neither $\mathbb{H}$ nor $\mathbb{O}$ is commutative. $\mathbb{O}$ is not even associative (!) but still at least alternative: any subalgebra generated by two elements (in particular $\mathbb{H}$) is associative.

The Lie groups $Spin(9)$ and $Spin(7)$

The Lie group $Spin(9)$ is the subgroup of $SO(16)$ generated by

$$\left\{ \begin{pmatrix} R_r & R_u \\ R_{\overline{u}} & -R_r \end{pmatrix}; r \in \mathbb{R}, u \in \mathbb{O}, r^2 + |u|^2 = 1 \right\}, \text{ where } R_u(v) = vu.$$

$Spin(7)$ is the stabilizer of $(1, 0)$ in $Spin(9)$. It is generated by

$$\{\operatorname{diag}(-R_u \circ L_u, L_u); u \in \operatorname{Im}\mathbb{O}\}, \text{ where } L_u(v) = uv.$$

Octonion-valued forms

The *real* Kähler 2-form on $\mathbb{C}^n$ is usually given by $\omega = \frac{i}{2}\sum_{j=1}^n dz_j \wedge \overline{dz_j}$, i.e. in terms of the *complex* coordinate 1-forms. In other words, it is regarded as a (very special) element of the algebra $\mathbb{C} \otimes \bigwedge^\bullet(\mathbb{C}^n)^*$.

In this manner we define the algebra of **octonion-valued forms** on a vector space V as $\bigwedge^\bullet_{\mathbb{O}} V^* := \mathbb{O} \otimes \bigwedge^\bullet V^*$. It is naturally equipped with the involution and the 'wedge' product $u\varphi \wedge v\psi := (uv)\varphi \wedge \psi$. For instance, the octonionic coordinates dx, dy on $\mathbb{O}^2$ belong to $\bigwedge^\bullet_{\mathbb{O}}(\mathbb{O}^2)^*$.

The canonical $Spin(9)$-invariant 8-form Ψ

Over the decades, a variety of integral and algebraic formulas for Ψ has been presented [Brown & Gray, Berger, Abe & Matsubara, Castrillón L'opez et al., Parton & Piccinni]. We give a new explicit formula in terms of octonion-valued forms. Let us denote

$$\Psi_{40} := ((\overline{dx} \wedge dx) \wedge \overline{dx}) \wedge dx, \quad \Psi_{31} := ((\overline{dy} \wedge dx) \wedge \overline{dx}) \wedge dx,$$
$$\Psi_{13} := ((\overline{dx} \wedge dy) \wedge \overline{dy}) \wedge dy, \quad \Psi_{04} := ((\overline{dy} \wedge dy) \wedge \overline{dy}) \wedge dy.$$

Theorem [JK]. *The (octonion-valued) form*

$$\Psi_{40} \wedge \overline{\Psi_{40}} + 4\,\Psi_{31} \wedge \overline{\Psi_{31}} - 5\left(\Psi_{31} \wedge \Psi_{13} + \overline{\Psi_{13}} \wedge \overline{\Psi_{31}}\right) + 4\,\Psi_{13} \wedge \overline{\Psi_{13}} + \Psi_{04} \wedge \overline{\Psi_{04}}$$

is a non-trivial real multiple of the $Spin(9)$-invariant 8-form Ψ.

This and analogical expressions for Ω, ϕ and Φ can be found in [1].

Valuations on convex bodies

Let $\mathcal{K}$ denote the family of non-empty compact convex sets in $\mathbb{R}^n$. A functional $\mu : \mathcal{K} \to \mathbb{C}$ is called a **valuation** if

$$\mu(K \cup L) = \mu(K) + \mu(L) - \mu(K \cap L)$$

whenever $K, L, K \cup L \in \mathcal{K}$. The space of translation-invariant and continuous (w.r.t. the Hausdorff metric) valuations is denoted by Val.

Let G be a compact subgroup of $SO(n)$ and let Val^G denote the subspace of G-invariant elements in Val. Then

Theorem [Alesker]. $\dim \mathrm{Val}^G < \infty$ *if and only if G acts transitively on the unit sphere S^{n-1}.*

Val^G has been described exhaustively for $SO(n)$, $U(n)$, $SU(n)$, $Sp(2)Sp(1)$, G_2 and $Spin(7)$ [Hadwiger, Alesker, Bernig, Fu, Park, Solanes et al.]. In the other cases, description of Val^G remains an open problem. There is an extremely rich algebraic structure on Val^G [Alesker, Bernig, Fu et al.]. The first task, however, is to get a basis.

$Spin(9)$-invariant valuations

Consider the case $G = Spin(9)$ acting on $\mathbb{R}^{16} \cong \mathbb{O}^2$. $\dim \mathrm{Val}^G = 143$ [Bernig & Voide]. How to find a basis?

First, any $\mu \in \mathrm{Val}^{Spin(9)}$ is *smooth* [Alesker], i.e. $\mu(\cdot) = \lambda\,\mathrm{vol}_{16}(\cdot) + \int_{nc(\cdot)} \tau$ for some $\lambda \in \mathbb{C}$ and a $\overline{Spin(9)}$-invariant differential 15-form τ on the sphere bundle $S\mathbb{O}^2$. This transforms the problem to the search for appropriate invariant differential forms.

However, there are more such forms than desired. In fact [Bernig, Bröcker], $\int_{nc(\cdot)} \tau = 0$ precisely when $D\tau = 0$ and $\pi^*\tau = 0$. Here $\pi : S\mathbb{O}^2 \to \mathbb{O}^2$ is the projection. To define the *Rumin differential* $D\tau$, recall the canonical contact 1-form α on $S\mathbb{O}^2$: $\alpha_{(x,v)}(w) := \langle v, d\pi(w)\rangle$ and that a form is called vertical if it is a multiple of α. Then there is a unique vertical 15-form ξ such that $D\tau := d(\tau + \xi)$ is vertical.

Further, since $\overline{Spin(9)}$ acts transitively on $S\mathbb{O}^2$, any invariant *differential* form is determined in a point, say $(0, (1, 0))$, i.e. by a $Spin(7)$-invariant form on the tangent space $T_{(0,(1,0))}S\mathbb{O}^2 \cong \mathbb{R}v \oplus T \oplus T$, where $T = \operatorname{Im}\mathbb{O} \oplus \mathbb{O}$. After factorizing the vertical forms out (for they are vanished by D), all what we need is $[\bigwedge^{15}(T \oplus T)^*]^{Spin(7)}$.

$Spin(7)$-invariant forms

It turns out that this space is built of the same kind of 'building blocks' as the canonical invariant 8-form above!

Theorem [JK, TW]. *The algebra $[\bigwedge^\bullet(T \oplus T)^*]^{Spin(7)}$ is generated by 96 (real) elements of $\bigwedge^\bullet_{\mathbb{O}}(T \oplus T)^*$. All the generators are composed entirely of the octonionic coordinates $\theta_1, \theta_2, \omega_1, \omega_2$ on $T \oplus T$.*

In order to compute the Rumin differential, one needs to know the exterior derivative of the respective differential forms of course. Let us regard a $\overline{Spin(9)}$-invariant differential form on $S\mathbb{O}^2$ as the formal expression of the corresponding $Spin(7)$-invariant alternating form on $T \oplus T$ in terms of $\theta_1, \theta_2, \omega_1, \omega_2$. Then

Theorem [JK, TW]. *The exterior derivative of a $\overline{Spin(9)}$-invariant differential form on $S\mathbb{O}^2$ is obtained applying the anti-derivation property of d together with the following rules:*

$$d\theta_1 = -\omega_1 \wedge \theta_1 - \theta_2 \wedge \overline{\omega_2}, \qquad d\omega_1 = -\omega_1 \wedge \omega_1 - \omega_2 \wedge \overline{\omega_2},$$
$$d\theta_2 = \theta_1 \wedge \omega_2 - \omega_1 \wedge \theta_2 + \theta_2 \wedge \omega_1, \qquad d\omega_2 = \omega_2 \wedge \omega_1.$$

[1] J. Kotrbatý, *Octonion-valued forms and the canonical 8-form on Riemannian manifolds with a Spin(9)-structure*, available at `arXiv:1808.02452`.