A discrete Borell-Brascamp-Lieb inequality.

D. Iglesias López (david.iglesias@um.es) (joint work with J. Yepes Nicolás (jesus.yepes@um.es))

Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain

Abstract

If $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ are non-negative measurable functions such that $h(x+y)$ is greater than or equal to the p-sum of $f(x)$ and $g(y)$, where $-1/n \leq p \leq +\infty$, $p \neq 0$, then the Borell-Brascamp-Lieb inequality asserts that the integral of h is not smaller than the q-sum of the integrals of f and g, for $q = p/(np+1)$.
Here we show a discrete analog of it for the sum over finite subsets of the integer lattice $\mathbb{Z}^n$: under the same assumption as before, for $A, B \subset \mathbb{Z}^n$, then $\sum_{A+B} h \geq [(\sum_{\mathrm{r}_f(A)} f)^q + (\sum_B g)^q]^{1/q}$, where $\mathrm{r}_f(A)$ is obtained by removing points from A in a particular way, and depending on f. We also prove that the classical Borell-Brascamp-Lieb inequality for Riemann integrable functions can be obtained as a consequence of this new discrete version.

1 Notation

As usual, we write $\mathbb{R}^n$ to represent the n-dimensional Euclidean space. The n-dimensional volume of a compact set $K \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(K)$ (when integrating, as usual, $\mathrm{d}x$ will stand for $\mathrm{dvol}(x)$), and as a discrete counterpart, we use $|A|$ to represent the cardinality of a finite subset $A \subset \mathbb{R}^n$.
We write $\pi_{i_1,\dots,i_k}$, $1 \leq i_1, \dots, i_k \leq n$, to denote the orthogonal projection onto the k-dimensional coordinate plane

$$\mathbb{R}e_{i_1} + \cdots + \mathbb{R}e_{i_k}.$$

Let $\mathbb{Z}^n$ be the integer lattice, i.e., the lattice of all points with integral coordinates in $\mathbb{R}^n$, and let $\mathbb{Z}^n_+ = \{x \in \mathbb{Z}^n : x_i \geq 0\}$.

2 The classical Borell-Brascamp-Lieb inequality

Relating the volume with the Minkowski addition of compact sets, one is led to the famous *Brunn-Minkowski inequality*. One form of it states that if $K, L \subset \mathbb{R}^n$ are compact, then

$$\mathrm{vol}(K+L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}, \tag{1}$$

with equality, if $\mathrm{vol}(K)\mathrm{vol}(L) > 0$, if and only if K and L are homothetic compact convex sets. Here $+$ is used for the *Minkowski* (vectorial) sum, i.e.,

$$A + B = \{a + b : a \in A, b \in B\}$$

for any $A, B \subset \mathbb{R}^n$.
The Brunn-Minkowski inequality is one of the most powerful theorems in Convex Geometry and beyond: it implies, among others, strong results such as the isoperimetric and Urysohn inequalities (see e.g. [7, s. 7.2]) or even the Aleksandrov-Fenchel inequality (see e.g. [7, s. 7.3]). It would not be possible to collect here all references regarding versions, applications and/or generalizations on the Brunn-Minkowski inequality. So, for extensive and beautiful surveys on them we refer the reader to [3].

Regarding an analytical counterpart for functions of the Brunn-Minkowski inequality, one is naturally led to the so-called *Borell-Brascamp-Lieb inequality*, originally proved in [1] and [2]. In order to introduce it, we first recall the definition of the p-sum of two non-negative numbers.

Definition 1 (p-sum) *Let $p \neq 0$ be a parameter varying in $\mathbb{R} \cup \{\pm\infty\}$. If $p \in \mathbb{R}$, with $p \neq 0$, then we set the p-sum of two non-negative numbers $a, b > 0$ as*

$$\mathcal{S}_p(a, b) = (a^p + b^p)^{1/p}.$$

For $p = \pm\infty$ we set $\mathcal{S}_{+\infty}(a, b) = \max\{a, b\}$ and $\mathcal{S}_{-\infty}(x, y) = \min\{a, b\}$. Finally, if $ab = 0$, we define $\mathcal{S}_p(a, b) = 0$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$, $p \neq 0$.

Note that $\mathcal{S}_p(a, b) = 0$, if $ab = 0$, is redundant for all $p < 0$, however it is relevant for $p > 0$. For a general reference for p-sums of non-negative numbers, we refer the reader to the classic text of Hardy, Littlewood, and Pólya [4].
The following theorem contains the Borell-Brascamp-Lieb inequality (see also [3] for a detailed presentation) which, as previously stated, can be regarded as the functional counterpart of the Brunn-Minkowski inequality. In fact, a straightforward proof of (1) can be obtained by applying (2) to the characteristic functions $f = \chi_{_K}$, $g = \chi_{_L}$ and $h = \chi_{_{K+L}}$ with $p = +\infty$.

Theorem 1 (The Borell-Brascamp-Lieb inequality) *Let $-1/n \leq p \leq +\infty$, $p \neq 0$, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that*

$$h(x+y) \geq \mathcal{S}_p\big(f(x), g(y)\big)$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \mathcal{S}_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right). \tag{2}$$

3 Working with finite sets

Next we move to the discrete setting, i.e., we consider finite subsets of (integer) points. We would like to point out that one cannot expect to obtain a discrete analog of the Borell-Brascamp-Lieb inequality just by "replacing integrals by sums" since it is not even possible to get a Brunn-Minkowski inequality in its classical form for the cardinality. Indeed, simply taking $A = \{0\}$ to be the origin and any finite set $B \subset \mathbb{Z}^n$, then

$$|A+B|^{1/n} < |A|^{1/n} + |B|^{1/n}.$$

Therefore, discrete counterparts for both the Brunn-Minkowski inequality and the Borell-Brascamp-Lieb inequality should either have a different structure or involve modifications of the sets.
An example of a Brunn-Minkowski type inequality with a modified structure could be

$$|A+B| \geq |A| + |B| - 1$$

for finite subsets A, B in $\mathbb{Z}^n$, which provides, in particular, a 1-dimensional discrete Brunn-Minkowski inequality.

3.1 Reducing one of the sets

An alternative to get a "classical" Brunn-Minkowski type inequality might be to transform (one of) the sets involved in the problem, either by adding or removing some points. In this spirit, two (equivalent) new discrete Brunn-Minkowski type inequalities have been obtained in [5]. Similarly, and in the case of removing points from the original (finite) set $A \subset \mathbb{Z}^n$, $A \neq \emptyset$, we may define a new set $\mathrm{r}_f(A)$ to *reduce* it according to a particular function f.
To this aim, we need the following notation. If $\Lambda \subset \mathbb{Z}^k$ is finite, $k \in \{1, \dots, n\}$, for each $m \in \mathbb{Z}$ we write $\Lambda(m)$ to represent the "section of Λ at m orthogonal to the (last) coordinate line $\mathbb{R}e_k$", i.e.,

$$\Lambda(m) = \{p \in \mathbb{Z}^{k-1} : (p, m) \in \Lambda\}.$$

Next, given a non-negative function $f : \Lambda \longrightarrow \mathbb{R}_{\geq 0}$ (which will be often referred to as a *weight function*), let $m_0 = m_0(\Lambda, f) \in \pi_k(\Lambda)$ be such that $\sum_{x \in \Lambda(m_0)} f(x, m_0) = \max_m \sum_{x \in \Lambda(m)} f(x, m)$. Certainly the integer m_0 providing the "maximum section" with respect to the weight function f does not have to be necessarily unique. In that case, we can establish as a criterion to take

$$m_0 = \max\left\{m' \in \pi_k(\Lambda) : \sum_{x \in \Lambda(m')} f(x, m') = \max_m \sum_{x \in \Lambda(m)} f(x, m)\right\}.$$

Now we define the function

$$\rho_k : \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\} \longrightarrow \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\}$$

given by

$$\rho_k(\Lambda) = \begin{cases} \Lambda \setminus \{m_0\} & \text{if } k = 1, \\ \Lambda \setminus \big(\Lambda(m_0) \times \{m_0\}\big) & \text{if } k > 1; \end{cases}$$

i.e., ρ_k acts on Λ just removing the "maximum section" $\Lambda(m_0)$, with respect to the weight function f, from the set. To complete the picture we set $\rho_k(\emptyset) = \emptyset$.
Then, for $1 \leq k < n$, we write

$$A_k^- = \bigcup_{m \in \pi_{n,\dots,k+1}(A_{k-1}^-)} \Big(\rho_k\big(A_{k-1}^-(m)\big) \times \{m\}\Big),$$

with $A_0^- = A$. Then we define

$$\mathrm{r}_f(A) = \rho_n\big(A_{n-1}^-\big).$$

In other words, $\mathrm{r}_f(A)$ is given by

$$\mathrm{r}_f(A) = \bigcup_{m \in \pi_n(A) \setminus \{m_0(A_{n-1}^-, f)\}} \Big(\mathrm{r}_f\big(A(m)\big) \times \{m\}\Big).$$

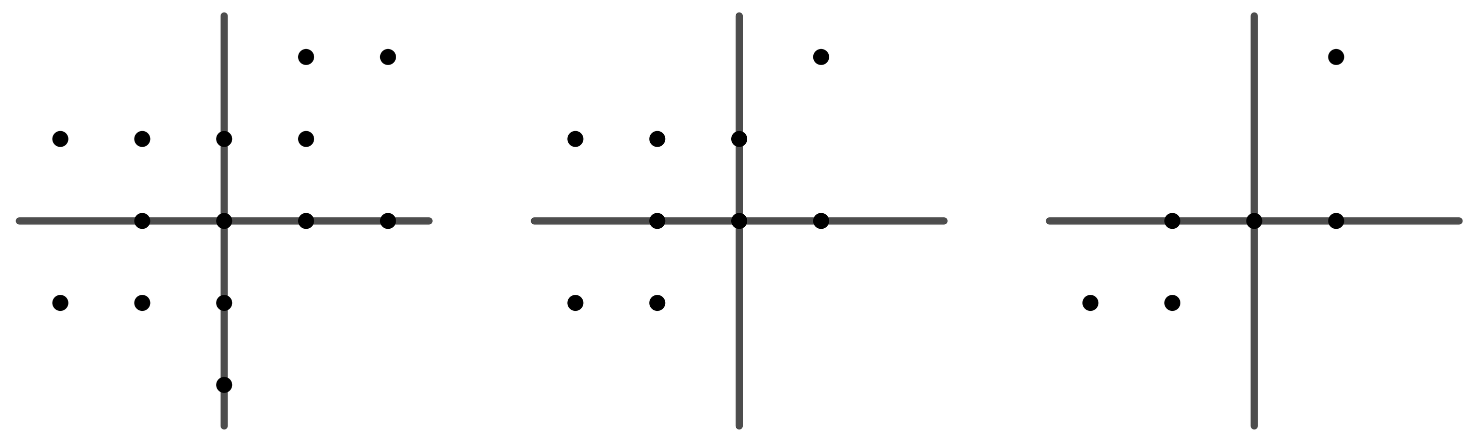

Figure 1: An example of transforming a discrete set A (left) into $\mathrm{r}_f(A)$ (right) when $f(x) = 1$ for all $x \in \mathbb{Z}^2$.

Using this technique, in [5, Theorem 2.2] the following result was shown, where $\varphi : \mathbb{Z}^n \longrightarrow \mathbb{R}_{\geq 0}$ is the constant weight function given by $\varphi(x) = 1$ for all $x \in \mathbb{Z}^n$.

Theorem 2 *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then*

$$|A+B|^{1/n} \geq \big|\mathrm{r}_\varphi(A)\big|^{1/n} + |B|^{1/n}. \tag{3}$$

The inequality is sharp.

Equality holds in (3) when both A and B are *lattice cubes*. By a lattice cube we mean the intersection of a cube $r[0,1]^n$, $r \in \mathbb{N}$, with the lattice $\mathbb{Z}^n$.

4 The main results

Our main goal is to show a discrete analog of Theorem 1, in the spirit of the above Theorem 2 for the classical Brunn-Minkowski inequality. These results can be found at [6].

Theorem 3 *Let $A, B \subset \mathbb{Z}^n$ be finite sets. Let $-1/n \leq p \leq +\infty$, $p \neq 0$, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h(x+y) \geq \mathcal{S}_p\big(f(x), g(y)\big)$$

for all $x \in A$, $y \in B$. Then

$$\sum_{z \in A+B} h(z) \geq \mathcal{S}_{\frac{p}{np+1}}\left(\sum_{x \in \mathrm{r}_f(A)} f(x), \sum_{y \in B} g(y)\right). \tag{4}$$

We note that the above result holds true for finite subsets $A, B \subset \mathbb{R}^n$, just suitably constructing the set $\mathrm{r}_f(A)$. We state Theorem 3 in the case of $\mathbb{Z}^n$ for the sake of simplicity.
As in the continuous setting, inequality (4) can be seen as a functional extension of the discrete Brunn-Minkowski inequality (3), just by considering the characteristic functions $f = \chi_{_A}$, $g = \chi_{_B}$ and $h = \chi_{_{A+B}}$, and taking $p = +\infty$.
Moreover, we also show that the classical Borell-Brascamp-Lieb inequality (2) can be obtained from the discrete version (4) under the mild (but necessary) assumption that the functions f, g are Riemann integrable:

Theorem 4 *The discrete Borell-Brascamp-Lieb inequality (4) implies the classical Borell-Brascamp-Lieb inequality (2), provided that the functions f, g are Riemann integrable.*

References

[1] C. Borell, Convex set functions in d-space, *Period. Math. Hungar.* **6** (1975), 111–136.

[2] H. J. Brascamp, E. H. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions and with an application to the diffusion equation, *J. Functional Analysis* **22** (4) (1976), 366–389.

[3] R. J. Gardner, The Brunn-Minkowski inequality, *Bull. Amer. Math. Soc.* **39** (3) (2002), 355–405.

[4] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*. Cambridge Mathematical Library, Reprint of the 1952 edition, Cambridge University Press, Cambridge, 1988.

[5] M. A. Hernández Cifre, D. Iglesias, J. Yepes Nicolás, On a discrete Brunn-Minkowski type inequality, *To appear in SIAM J. Discrete Math.*.

[6] D. Iglesias, J. Yepes Nicolás, On discrete Borell-Brascamp-Lieb inequalities, *Submitted*.

[7] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*. Second edition. Cambridge University Press, Cambridge, 2014.