Integral Geometry and Valuations

Gil Solanes

Serra Húnter Fellow
Departament de Matemàtiques
Universitat Autònoma de Barcelona

Valuations

Definition

$\varphi\colon \mathcal{K}^n \to \mathbb{R}$ is a **valuation** if

$$\varphi(A \cup B) = \varphi(A) + \varphi(B) - \varphi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}^n$.
Val $= \{$*translation-invariant, continuous valuations*$\}$.

Proposition (McMullen)

$$\mathrm{Val} = \bigoplus_{k=0}^{n} \bigoplus_{\epsilon=\pm} \mathrm{Val}_k^{\epsilon}$$

$$\mathrm{Val}_n = \langle \mathrm{vol} \rangle, \qquad \mathrm{Val}_0 = \langle \chi \rangle.$$

Examples

Given $B \in \mathcal{K}^n$

$$\begin{aligned} \mathrm{vol}_B(A) :&= \mathrm{vol}(A+B) \\ &= \sum_{j=0}^{n} \binom{n}{j} V(A[j], B[n-j]) \\ &= \int_{\mathbb{R}^n} \chi(A \cap (x-B))dx \end{aligned}$$

Intrinsic volumes

Let B^n =unit ball, $\omega_n = \mathrm{vol}(\mathrm{B}^n)$

$$\begin{aligned} \mu_j(A) :&= \frac{1}{\omega_{n-j}} \binom{n}{j} V(A[j], \mathrm{B}^n[n-j]) \\ &= \int_{N(A)} \kappa_j \end{aligned}$$

for canonical $\kappa_j \in \Omega^{n-1}(S\mathbb{R}^n)$.

Smooth valuations

Definition

A valuation $\varphi \in \mathrm{Val}$ is called **smooth** if

$$\varphi(A) = \int_{N(A)} \omega + \lambda \mathrm{vol}$$

for some fixed translation-invariant form $\omega \in \Omega^{n-1}(S\mathbb{R}^n)$.

Denote

$$\mathrm{Val}^\infty := \left\langle \int_{N(\cdot)} \omega \right\rangle_\omega \oplus \langle \mathrm{vol} \rangle.$$

Smooth valuations

Definition

A valuation $\varphi \in \mathrm{Val}$ is called **smooth** if

$$\varphi(A) = \int_{N(A)} \omega + \lambda \mathrm{vol}$$

for some fixed translation-invariant form $\omega \in \Omega^{n-1}(S\mathbb{R}^n)$.

Denote

$$\mathrm{Val}^\infty := \left\langle \int_{N(\cdot)} \omega \right\rangle_\omega \oplus \langle \mathrm{vol} \rangle.$$

Crofton formulas

$$\mu_i(A) = \int_{\mathrm{Gr}_i} \mathrm{vol}(\pi_E(A))dE$$
$$= \int_{\mathrm{Gr}_i}\int_E \chi(A\cap(x+E^\perp))dxdE$$
$$= \int_{\overline{\mathrm{Gr}}_{n-i}} \chi(A\cap F)dF$$

Here

$$dE(\mathrm{Gr}_i) = \binom{n}{i}\frac{\omega_n}{\omega_i\omega_{n-i}} =: \begin{bmatrix} n \\ i \end{bmatrix}$$

Reproductive property

$$\begin{bmatrix} i+j \\ i \end{bmatrix}\mu_{i+j} = \int_{\overline{\mathrm{Gr}}_{n-i}} \mu_j(\pi_E(A))dE$$

Crofton valuations

Definition

Given a signed mesure m in Gr_i define $\mathrm{Cr}(m) \in \mathrm{Val}_i^+$ by

$$\mathrm{Cr}(m)(A) = \int_{\mathrm{Gr}_i} \mathrm{vol}(\pi_E(A)) d_E m$$

Klain function

Theorem (Klain)

Each valuation $\varphi \in \mathrm{Val}_k^+$ *is characterized by its* **Klain function** $\mathrm{Kl}_\varphi \colon \mathrm{Gr}_k \to \mathbb{R}$

$$\varphi(A) = \mathrm{Kl}_\varphi(E)\mathrm{vol}_k(A) \qquad \forall A \subset E \in \mathrm{Gr}_k$$

Proposition

i) $\mathrm{Kl}_{(\mathrm{vol} B)_i}(E^\perp) = \mathrm{vol}_{n-i}(\pi_E B), \qquad E \in \mathrm{Gr}_{n-i}, \quad B = -B$

ii) $\mathrm{Kl}_{\mathrm{Cr}(m)}(E) = \int_{\mathrm{Gr}_i} \cos(E, F) d_F m \qquad$ (cosine transform)

Corollary (Hadwiger)

$$\mathrm{Val}^{\mathrm{SO}(n)} = \langle \mu_0, \ldots, \mu_n \rangle_{\mathbb{R}}$$

Kinematic formulas

Corollary

Kinematic formulas

$$\int_{\overline{\mathrm{SO(n)}}} \chi(A \cap gB)dg = \sum_{i=0}^{n} c_i \mu_i(A)\mu_{n-i}(B)$$

More generally

$$\int_{\overline{\mathrm{SO(n)}}} \mu_k(A \cap gB)dg = \sum_{i=k}^{n} c_{i,k} \mu_{i+k}(A)\mu_{n-i}(B)$$

Additive kinematic formulas:

$$\int_{\mathrm{SO(n)}} \mu_{n-k}(A + hB)dh = \sum_{i=k}^{n} c_{i,k} \mu_{n-i-k}(A)\mu_i(B)$$

Irreducibility theorem

Recall

$$\mathrm{Val} = \bigoplus_{i=0}^{n} \bigoplus_{\varepsilon=\pm} \mathrm{Val}_i^{\varepsilon}$$

Theorem (Alesker)

Each component in McMullen's decomposition is an irreducible representation of $\mathrm{GL}(n,\mathbb{R})$*.*
i.e. $\mathrm{Val}_i^{\epsilon}$ *has no closed proper* $\mathrm{GL}(n,\mathbb{R})$*-invariant subspace.*

Corollary

i) $\langle \mathrm{vol}_A \rangle_A$ *is dense in* Val

ii) $im(\mathrm{Cr})$ *is dense in* Val^{+}

iii) Val^{∞} *is dense in* Val

Isotropic spaces

Suppose $G \subset \mathrm{O}(n)$ acts transitively on S^{n-1}. Then

i) Val^G admits a finite basis $\varphi_1, \ldots, \varphi_N$ (smooth)

ii) $\forall \varphi \in \mathrm{Val}^G\ \exists c_{i,j}$ such that

$$\int_{\overline{G}} \varphi(A \cap gB) dg = \sum_{i,j=1}^{N} c_{i,j} \varphi_i(A) \varphi_j(B)$$

These formulas are encoded by $k\colon \mathrm{Val}^G \to \mathrm{Val}^G \otimes \mathrm{Val}^G$.

iii) $\exists a\colon \mathrm{Val}^G \to \mathrm{Val}^G \otimes \mathrm{Val}^G$ such that

$$\int_G \varphi(A + hB) dh = a(\varphi)(A, B)$$

iv) Val^G is spanned by $\mathrm{vol}_A^G := k(\chi)(A) = a(\mathrm{vol})(A)$.

Classification of affine isotropic spaces:

$\mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Sp}(n)\mathrm{U}(1), \mathrm{Sp}(n)\mathrm{Sp}(1), \mathrm{G}_2, \mathrm{Spin}(7), \mathrm{Spin}(9)$

Alesker product

Theorem (Alesker)

There exists a continuous commutative product $\mathrm{Val}^\infty \times \mathrm{Val}^\infty \to \mathrm{Val}^\infty$ *such that*

$$\mathrm{vol}_A \cdot \mathrm{vol}_B(C) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi(C \cap (x - A) \cap (y - B)) dxdy.$$

It fulfills:

i)
$$\varphi \cdot \mathrm{vol}_A = \int_{\mathbb{R}^n} \varphi(\cdot \cap (x - A)) dx.$$

ii) $\chi = 1$.

iii) $\varphi \in \mathrm{Val}_i, \phi \in \mathrm{Val}_j \Longrightarrow \varphi \cdot \phi \in \mathrm{Val}_{i+j}$.

Example

$$\begin{aligned}\mu_i \cdot \mu_j(A) &= \mu_i \cdot \int_{\overline{\mathrm{Gr}}_{n-j}} \chi(A \cap F) dF \\ &= \int_{\overline{\mathrm{Gr}}_{n-j}} \mu_i(A \cap F) dF \\ &= \begin{bmatrix} i+j \\ j \end{bmatrix} \mu_{i+j}(A)\end{aligned}$$

Alesker-Poincaré pairing

Proposition

The pairing $\mathrm{pd}\colon \mathrm{Val}^\infty \times \mathrm{Val}^\infty \to \mathbb{R}$

$$\mathrm{pd}(\varphi, \mu) = (\varphi \cdot \mu)_n = \lim_{r\to\infty} \frac{\varphi \cdot \mu(r\mathrm{B}^n)}{\mathrm{vol}(r\mathrm{B}^n)}$$

is a nondegenerate bilinear form. In particular the induced map

$$\mathrm{pd}\colon \mathrm{Val}^G \to \mathrm{Val}^{G*}$$

is an isomorphism.

Example: $\varphi = \mathrm{Cr}(m) \in \mathrm{Val}_i^{+,\infty}, \phi \in \mathrm{Val}_{n-i}^\infty$

$$\mathrm{pd}(\varphi, \phi) = \int_{\mathrm{Gr}_i} \mathrm{Kl}_\phi(E^\perp) d_E m$$

Fundamental theorem

Theorem

Considering $k(\chi) \in \mathrm{Val}^G \otimes \mathrm{Val}^G = \mathrm{Hom}(\mathrm{Val}^{G*}, \mathrm{Val}^G)$*, and* $\mathrm{pd} \in \mathrm{Hom}(\mathrm{Val}^G, \mathrm{Val}^{G*})$*,*

$$k(\chi) = \mathrm{pd}^{-1}.$$

Proof.

Given $A \in \mathcal{K}$ consider $A \in (\mathrm{Val}^G)^*$. Given $\phi \in \mathrm{Val}^G$

$$\begin{aligned}\langle \mathrm{pd} \circ k(\chi)(A), \phi\rangle &= (\phi \cdot k(\chi)(A))_n \\ &= \lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R)} \int_{\overline{G}} \phi(B_R \cap gA)dg \\ &= \phi(A) \\ &= \langle A, \phi\rangle.\end{aligned}$$

□

Fourier transform

Proposition

There exists a map $\mathrm{F} : \mathrm{Val}_k^{+,\infty} \to \mathrm{Val}_{n-k}^{+,\infty}$, *called* **Fourier transform**, *such that*

$$\mathrm{Kl}_{\mathrm{F}(\varphi)}(E) = \mathrm{Kl}_\varphi(E^\perp)$$

Proof: Alesker-Bernstein: every $\varphi \in \mathrm{Val}_k^{\infty,+}$ has a Crofton measure m. Hence

$$\begin{aligned}\mathrm{Kl}_\varphi(E^\perp) &= \int_{\mathrm{Gr}_k \ni F} \cos(E^\perp, F) dm \\ &= \int_{\mathrm{Gr}_{n-k} \ni F^\perp} \cos(E, F^\perp) d\perp_* m\end{aligned}$$

where $\perp : \mathrm{Gr}_k \to \mathrm{Gr}_{n-k}$. □

Example: $F(\mu_i) = \mu_{n-i}$.

Convolution

Proposition (Bernig-Fu)

There exists a **convolution** $*\colon \mathrm{Val}^\infty \otimes \mathrm{Val}^\infty \to \mathrm{Val}^\infty$ *such that*

$$\mathrm{vol}_A * \mathrm{vol}_B = \mathrm{vol}_{A+B}$$

It fulfills:

i) $\varphi * \mathrm{vol}_A = \varphi(\cdot + A)$

ii) $\phi * \mathrm{vol} = \phi$

iii) $\varphi \in \mathrm{Val}_{n-i}, \phi \in \mathrm{Val}_{n-j} \implies \varphi * \phi \in \mathrm{Val}_{n-i-j}$.

Example: $\mu_{n-i} * \mu_{n-j} = \left[\begin{array}{c} i+j \\ j \end{array} \right] \mu_{n-i-j}$.

Indeed, $V(A_1, \ldots, A_{n-k}, \cdot) = \frac{(n-k)!}{n!} \frac{d^k}{dt_1 \cdots dt_k}\Big|_{t_i=0} \mathrm{vol}_{\sum t_i A_i}$

yields

$$V(\mathrm{B}[n-k], \cdot) * V(\mathrm{B}[n-l], \cdot) = \frac{k!l!}{n!(k+l-n)!} V(B[n-k-l], \cdot)$$

Theorem (Bernig-Fu)

$$\mathsf{F}(\varphi)\cdot\mathsf{F}(\psi)=\mathsf{F}(\varphi*\psi)$$

Proof. (Even case) Let $\varphi=\mathrm{vol}_A, \psi=\mathrm{vol}_B$ with A, B symmetric. Let $(\mathrm{vol}_B)_i=\mathrm{Cr}(m_{B,i})$. Then

$$\begin{aligned}(\mathsf{F}(\mathrm{vol}_A)\cdot\mathsf{F}(\mathrm{vol}_B))_n &= \sum_i(\mathsf{F}\,\mathrm{vol}_A)_i\cdot\mathsf{F}((\mathrm{vol}_B)_i)\\ &= \sum_i\int_{\mathrm{Gr}_{n-i}}\mathrm{Kl}_{(F\mathrm{vol}_A)_i}(E^\perp)d_{E^\perp *}m_{B,i}\cdot\mathrm{vol}\\ &= \sum_i\int_{\mathrm{Gr}_i}\mathrm{Kl}_{(\mathrm{vol}_A)_{n-i}}(H^\perp)d_H m_{B,i}\cdot\mathrm{vol}\\ &= \sum_i\int_{\mathrm{Gr}_i}\mathrm{vol}_i(\pi_H A)d_H m_{B,i}\cdot\mathrm{vol}\\ &= \sum_i(\mu_B)_i(A)\mathrm{vol}=\mathrm{vol}(A+B)\mathrm{vol}\\ &= (\mathsf{F}\,\mathrm{vol}_{A+B})_n\end{aligned}$$

so degree n components agree. For lower degree components, compare Klain functions and use $(F\mathrm{vol}_A)|_E=F\mathrm{vol}_{\pi_E A}$.

Corollary

$$a(F\phi) = (F \otimes F)k(\phi)$$

Proof.

From

$$\mathrm{pd} \circ (F \otimes F) = \mathrm{pd}$$

one deduces

$$(F \otimes F) \circ k(\chi) = k(\chi) = a(\mathrm{vol})$$

Therefore

$$\begin{aligned} a(F\phi) &= (F\phi \otimes \mathrm{vol}) * a(\mathrm{vol}) \\ &= (F\phi \otimes F\chi) * (F \otimes F)k(\chi) \\ &= (F \otimes F)((\phi \otimes \chi) \cdot k(\chi)) \\ &= (F \otimes F)k(\phi) \end{aligned}$$

□

Minkowski spaces

Definition

Let $(\mathbb{R}^n, \|\cdot\|_B)$ be a finite dimensional normed space with smooth strictly convex unit ball B.
The Holmes-Thompson volume is

$$\mathrm{vol}_n^{HT}(A) = \frac{\mathrm{vol}_n(B^\circ)}{\omega_n}\mathrm{vol}_n(A)$$

Theorem (Schneider-Wieacker, Álvarez-Fernandes, Bernig)

There exist $\mu_1^B, \ldots, \mu_n^B \in \mathrm{Val}^{+,\infty}$ such that

i) μ_1^B *extends* $\| \ \|_B$

ii) μ_i^B *extends the* $i-$*dimensional Holmes-Thompson volume.*

iii) $\mu_i^B \cdot \mu_j^B = \begin{bmatrix} i+j \\ i \end{bmatrix} \frac{\omega_{i+j}}{\omega_i \omega_j} \mu_{i+j}^B.$

Proof.

If $A \subset E^\perp$ for some $E \in \mathrm{Gr}_k$, then

$$(\mathrm{vol}_{B^\circ})_{n-k}(A) = \mathrm{vol}_{n-k}(A)\mathrm{vol}_k(\pi_E(B^\circ)) \\ = \mathrm{vol}_{n-k}(A)\mathrm{vol}_k((E \cap B)^\circ)$$

Put $\mu_k^B := \omega_k^{-1}(\mathrm{F}\,\mathrm{vol}_{B^\circ})_k$. For $C \subset E \in \mathrm{Gr}_k$

$$\mu_k^B(C) = \mathrm{vol}_k(C)\frac{\mathrm{vol}_k((E \cap B)^\circ)}{\omega_k} = \mathrm{vol}_k^{HT}(C).$$

In particular μ_1^B extends $\|\cdot\|_B$.
The μ_k^B are multiplicative just as the euclidean μ_k because

$$\mathrm{F}(\mu_i^B \cdot \mu_j^B) = c(\mathrm{vol}_{B^\circ})_i * (\mathrm{vol}_{B^\circ})_j = c(\mathrm{vol}_{B^\circ})_{n-i-j} = c\,\mathrm{F}\,\mu_{i+j}^B$$

□

Orthogonal group

$V=\mathbb{R}^n, G=\mathrm{SO}(n)$ (Blaschke - Santaló - Federer - Chern)

For $i+j\leq n$

$$\mu_i\cdot\mu_j=c_{i,j}\mu_{i+j}$$
$$\mu_{n-i}*\mu_{n-j}=c_{i,j}\mu_{n-i-j}$$

In particular,

$$\mathrm{Val}^{\mathrm{SO}(n)}=\frac{\mathbb{R}[t]}{(t^{n+1})},\qquad t=\frac{2}{\pi}\mu_1$$

Unitary group

$V = \mathbb{C}^n, G = \mathrm{U}(n)$ (Alesker,Fu,Bernig-Fu)

Theorem (Alesker)

As an algebra, $\mathrm{Val}^{U(n)}$ *has two generators:*

$$t = \frac{2}{\pi}\mu_1$$

$$s = \int_{\mathrm{Gr}_1^{\mathbb{C}}} \mathrm{vol}_2(\pi_E)dE$$

Theorem (Fu)

$$\mathrm{Val}^{U(n)} = \frac{\mathbb{R}[t,s]}{(f_{n+1}(t,s), f_{n+2}(t,s))}$$

Monomials

$$t^i \cdot s^j = t^i \int_{\overline{\mathrm{Gr}}_{n-j}^{\mathbb{C}}} \chi(\cdot \cap \overline{F}) d\overline{F}$$
$$= \int_{\overline{\mathrm{Gr}}_{n-j}^{\mathbb{C}}} t^i(\cdot \cap \overline{F}) d\overline{F}$$

Its Fourier transform is

$$F(t^i \cdot s^j) = F(t^{2n-i}) * F(s^{n-j})$$
$$= t^{2n-i} * s^{n-j}$$
$$= t^{2n-i} * \int_{\mathrm{Gr}_{n-j}^{\mathbb{C}}} \mathrm{vol}_{2j}(\pi_F(\cdot)) dF$$
$$= \int_{\mathrm{Gr}_{n-j}^{\mathbb{C}}} t^{2n-i}(\pi_F(\cdot)) dF$$

Each $\varphi \in \mathrm{Val}^{U(n)}$ is smooth;

$$\text{i. e.} \quad \varphi = \int_{N(\cdot)} \omega + \lambda \,\mathrm{vol}, \qquad \omega \in \Omega^{2n-1}(SV)^{\overline{U(n)}}$$

The space $\omega \in \Omega^{2n-1}(SV)^{\overline{U(n)}}$ can be described using invariant theory
It is generated by (H.Park)

$$\alpha, \beta, \gamma \in \Omega^1(S\mathbb{C}^n), \qquad d\alpha, d\beta, d\gamma, \kappa \in \Omega^2(S\mathbb{C}^n)$$

where

$$\begin{aligned} \alpha_{(x,v)} &= \langle dx, v\rangle \\ \beta_{(x,v)} &= \langle dx, Jv\rangle \\ \gamma_{(x,v)} &= \langle dv, Jv\rangle \\ \kappa_{(x,v)} &= \langle dx, Jdx\rangle \end{aligned}$$

Hermitian intrinsic volumes

Proposition (Bernig-Fu)

$\mathrm{Val}^{U(n)}$ *admits a basis*

$$\mu_{k,q}, \qquad 0, k-n \leq q \leq k/2, \qquad 0 \leq k \leq 2n$$

such that

$$\mathrm{Kl}_{\mu_{k,q}}(\mathbb{C}^q \oplus \mathbb{R}^{k-2q}) = 1$$
$$\mathrm{Kl}_{\mu_{k,q}}(\mathbb{C}^r \oplus \mathbb{R}^{k-2r}) = 0, \qquad r \neq q$$

These valuations are called hermitian intrinsic volumes. Their Fourier transform is

$$F(\mu_{k,q}) = \mu_{2n-k,n-k+q}$$

Proposition (Bernig-Fu)

$$t \cdot \mu_{k,q} = \frac{\omega_{k+1}}{\pi \omega_k}((k-2q+1)\mu_{k+1,q} + 2(q+1)\mu_{k+1,q+1}).$$

Proof.

$$t \cdot \mu_{k,q} = F(t^{2n-1} * \mu_{2n-k,n-k+q})$$

and

$$\begin{aligned} t^{2n-1} * \varphi(A) &= \frac{d}{dr}\Big|_{r=0} \mathrm{vol}_{r\mathrm{B}_1} * \varphi(A) \\ &= \frac{d}{dr}\Big|_{r=0} \varphi(A + r\mathrm{B}_1) \end{aligned}$$

□

Proposition (Abardia-Gallego-S.)

$$s^j = \frac{j!}{\pi^j} \sum_{q=0}^{j} \frac{1}{4^{j-q}} \binom{2j-2q}{j-q} \mu_{2j,q}$$

Proof.

Apply convolution with $\frac{d}{dt}\mathrm{vol}_{tA}$ to both sides and check they agree $\forall A$.
RHS: Rumin operator.
LHS: let $C \subset \langle v \rangle$

$$\frac{d}{dt}\mathrm{vol}_{tA} * s(C) = \int_{N(C)} h_A \cdot \beta \wedge dS(v^\perp)$$

□

Proposition (Bernig-Fu)

$$\mu_{k,q} = p_{k,q}(s,t)$$

with explicit polynomials $p_{k,q} \in \mathbb{R}[s,t]$.

$$\mathrm{Val}^{\mathrm{U}(n)} = \frac{\mathbb{R}[s,t]}{(p_{n+1,0}, p_{n+2,0})}.$$

This leads to explicit kinematic formulas.

Valuations and curvature measures

Let M^n be a smooth manifold. A smooth valuation on M is a functional $\mu : \mathcal{P}(M) \to \mathbb{R}$ of the form

$$\mu(A) = \int_{N(A)} \kappa + \int_A \omega, \qquad A \in \mathcal{P}(M).$$

where

- $\mathcal{P}(M)$ is some class of sufficiently nice compact subsets $A \subset M$
- $N(A) \subset SM$ is the normal cycle of A,
- $\kappa \in \Omega^{n-1}(SM), \omega \in \Omega^n(M)$

A curvature measure is

$$\Phi(A, U) = \int_{N(A)\cap\pi^{-1}U} \kappa + \int_U \omega, \qquad A \in \mathcal{P}(M),\ U \subset M \text{ Borel.}$$

$\mathcal{V}(M)$ and $\mathcal{C}(M)$

Let

$$\mathcal{V}(M) = \{\mu \text{ valuation on } M\}$$
$$\mathcal{C}(M) = \{\Phi \text{ curvature measure on } M\}$$

Canonical globalization map glob: $\mathcal{C}(M) \to \mathcal{V}(M)$ given by

$$\mathrm{glob}(\Phi) = \Phi(\cdot, M).$$

$\mathcal{V}(M)$ is a commutative algebra with $1 = \chi$ (Chern: $\chi \in \mathcal{V}(M)$)
$\mathcal{C}(M)$ is a module over $\mathcal{V}(M)$.

Kinematic formulas in isotropic spaces

Let M^n be a manifold, and G a group of isometries acting transitively on SM.
Then $\dim(\mathcal{C}^G(M)) < \infty$.

Theorem (Fu'90)

Let $\Phi_1, \ldots, \Phi_m$ be a basis of $\mathcal{C}^G(X)$. Then

$$\int_G \Phi_k(A \cap gB, U \cap gV) dg = \sum_{i,j} c_{kij} \Phi_i(A, U) \Phi_j(B, V)$$

for certain constants c_{kij}.

Isotropic spaces

Classification of isotropic spaces:

- Affine isotropic spaces: $(\mathbb{R}^n, \overline{\mathrm{SO}(n)}), (\mathbb{C}^n, \overline{\mathrm{U}(n)}), \ldots$
- Rank one symmetric spaces:
  $S^n, H^n, \mathbb{C}P^n, \mathbb{C}H^n, \mathbb{H}P^n, \mathbb{H}H^n, \mathbb{O}P^2, \mathbb{O}H^2$.
- Exceptional spheres: $(S^6, G_2), (S^7, \mathrm{Spin}(7))$.

Howard's transfer principle

Local kinematic formulas depend only on the isotropy group.

Real space forms

$M = S^n, G = SO(n+1)$

$$\mathcal{V}(S^n)^G = \frac{\mathbb{R}[t]}{(t^{n+1})}$$

with $t = \mu_1|_M$, or $t = k(\chi)(S^{n-1})$.

Complex space forms

$M = \mathbb{C}P^n, G = \mathrm{Isom}(M)$ (Abardia-Gallego-S., Bernig-Fu-S.)

The algebra of invariant valuations $\mathcal{V}(\mathbb{C}P^n)^G$ has two generators:

$$s = k(\chi)(\mathbb{C}P^{n-1}) = \int_G \chi(\cdot \cap g\mathbb{C}P^{n-1})dg$$
$$t = i^*(\mu_1)$$

where $i\colon \mathbb{C}P^n \hookrightarrow \mathbb{R}^N$ is any isometric embedding (Weyl principle).

Unitary curvature measures

Invariant curvature measures:

$$\mathcal{C}(\mathbb{C}P^n)^G \equiv \mathcal{C}(\mathbb{C}^n)^{\overline{U(n)}} \equiv \langle B_{k,q}\rangle \oplus \langle \Gamma_{k,q}\rangle$$

In $\mathbb{C}^n$ hermitian intrinsic volumes have non-unique localizations:

$$\mu_{k,q} = \mathrm{glob}_{\mathbb{C}^n}(B_{k,q}) = \mathrm{glob}_{\mathbb{C}^n}(\Gamma_{k,q})$$

Hermitian intrinsic volumes in $\mathbb{C}P^n_\lambda$

$$\mu_{k,q} := \mathrm{glob}_{\mathbb{C}P^n}(B_{k,q})$$

Proposition

$$t = \sum_{k,q=0}^{\infty} \frac{\partial^{k+q} g(x,y)}{\partial x^k \partial y^q}\Big|_{(0,0)} \mu_{2k+2q+1,q}$$

with $g(x,y)$ explicit.

Proposition

$$s \cdot \mu_{k,q} = a_{k,q}\, \mu_{k+2,q} + b_{k,q}\, \mu_{k+2,q+1}$$

both in $\mathbb{C}^n$ and $\mathbb{C}P^n$ with the same $a_{k,q}, b_{k,q}$.

Algebra isomorphism

Theorem (Bernig-Fu-Solanes)

In $\mathbb{C}P^n$

$$\mu_{k,q} = (1-s)p_{k,q}(t\sqrt{1-s}, s).$$

Corollary

There exists an isomorphism *of algebras* $I : Val^{U(n)} \to \mathcal{V}(\mathbb{C}P^n)$ *given by* $I(s) = s, I(t) = t\sqrt{1-s}$.