Hollow lattice polytopes and convex geometry

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(mostly joint with O. Iglesias-Valiño)

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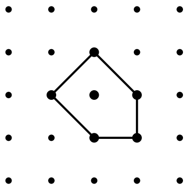

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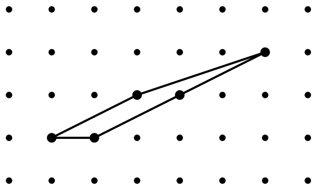

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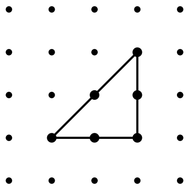

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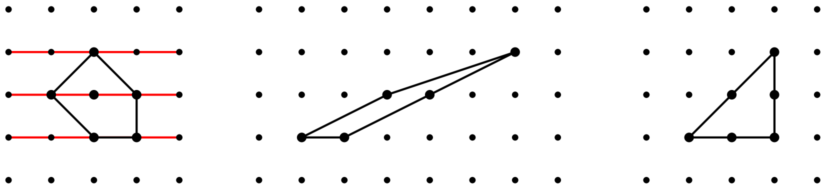

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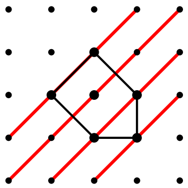

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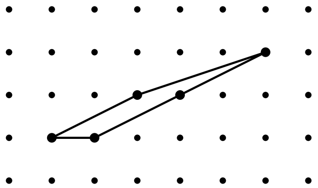

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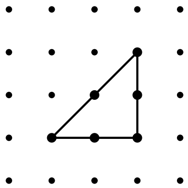

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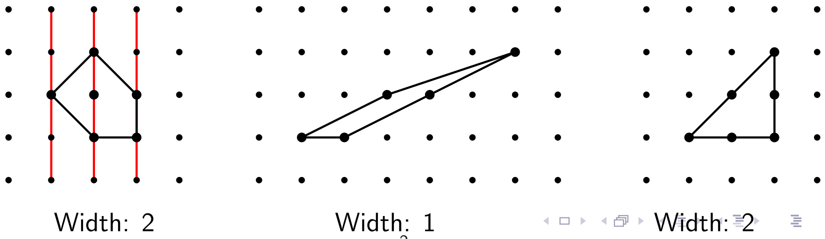

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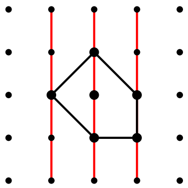

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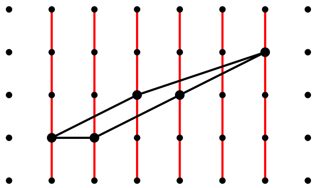

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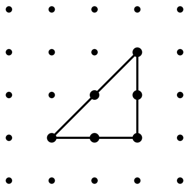

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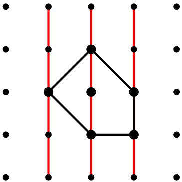

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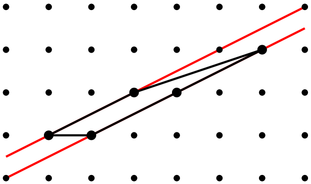

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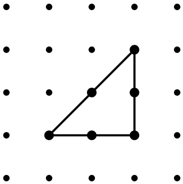

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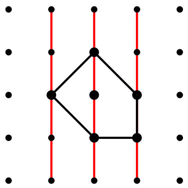

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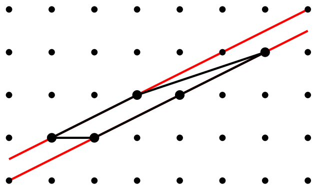

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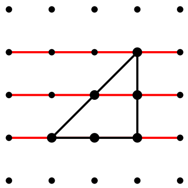

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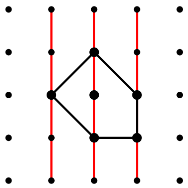

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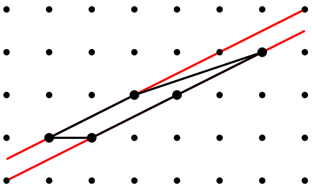

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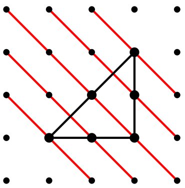

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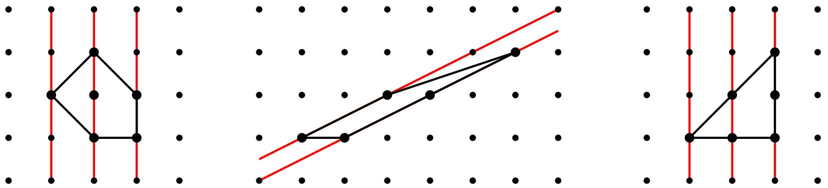

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For each dimension d,

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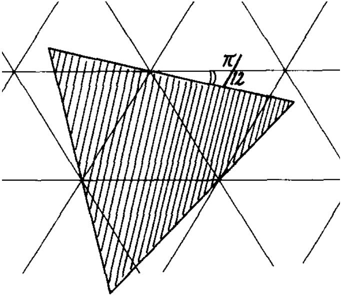
Known values: $W_1 = 1$, $W_2 = 1 + 3/\sqrt{2} \simeq 2.1547$ (Hurkens 1990)



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Current "guess": $W_d \in O(d)$ (perhaps modulo poly-log factors).

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Moreover, the limit is the same if restricted to lattice polytopes instead of arbitrary convex bodies.

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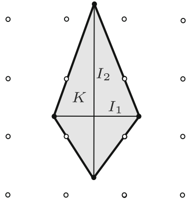
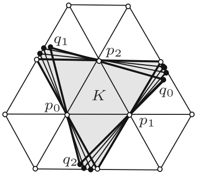
Theorem (Averkov-Wagner 2012)

Let K be a lattice-free convex 2-body with $w > 1$. Then

$$\mathrm{vol}(K) \leq \begin{cases} \frac{w^2}{2(w-1)} & \text{for } w \in (1,2], \\ \frac{3w^2}{3w+1-\sqrt{1+6w-3w^2}} & \text{for } w \in [2, 1+\frac{2}{\sqrt{3}}]. \end{cases}$$

The bound is attained iff K is as follows, respectively:



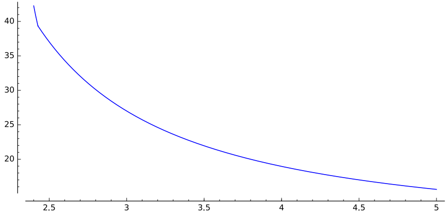


Width vs. volume, dim 3

Theorem (IglesiasValiño-Santos, 2018)

Let K be a lattice-free convex 3-body of lattice width $w > 1 + 2/\sqrt{3} = 2.155$. Then,

$$\mathrm{vol}(K) \leq \begin{cases} \frac{3w^3}{4(w-(1+2/\sqrt{3}))}, & \text{if } w \leq \frac{2}{\sqrt{3}}(\sqrt{5}-1)+1 = 2.427, \\ \frac{8w^3}{(w-1)^3}, & \text{if } w \geq 2.427. \end{cases}$$



These bound are *not* attained.

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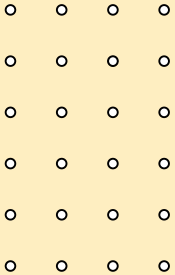

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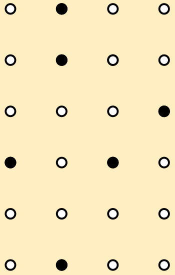

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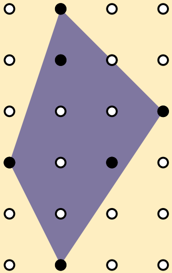

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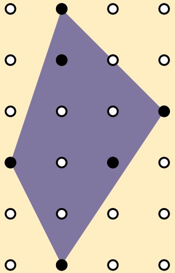

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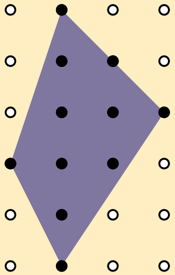

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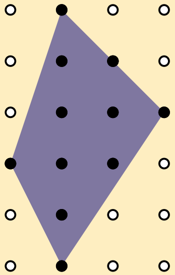

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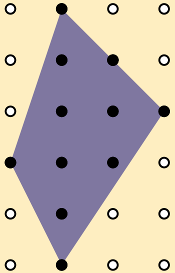
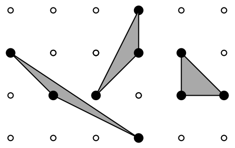
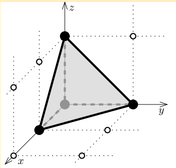
  E.g.: empty d-simplex $\Leftrightarrow$ lattice d-polytope with exacty $d+1$ lattice points.







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Remark

Volume, combinatorial type, hollowness, emptyness, width ... are invariant modulo unimodular equivalence.

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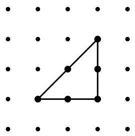

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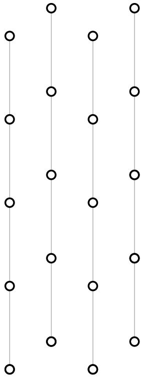

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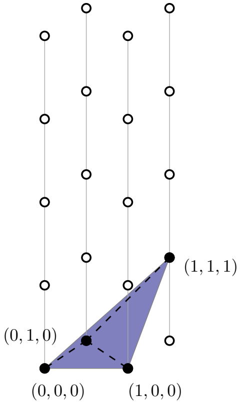

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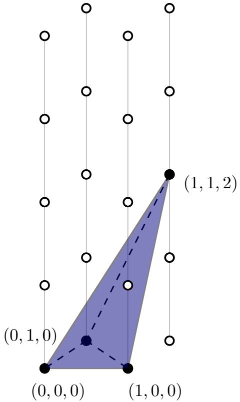

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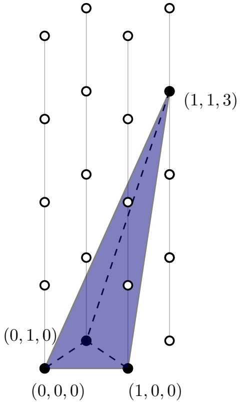

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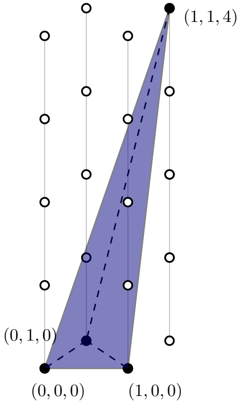

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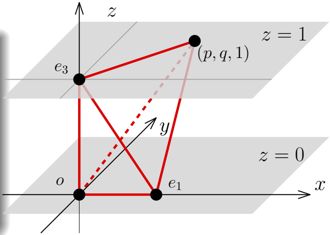

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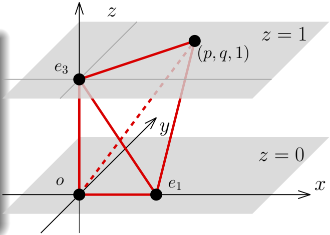

That is:

There are infinitely many empty tetrahedra, but they form a *two-parameter family* that we can describe completely.

Classification of hollow 3-polytopes

What about *hollow* 3-polytopes?

Theorem

The whole list of hollow 3-polytopes consists of:

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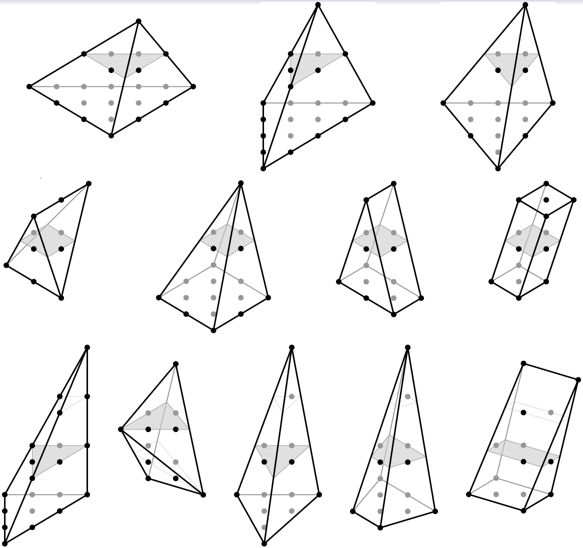
Remark

The three cases (1), (2) and (3) correspond to what is the minimal dimension of a lattice projection of P that is still hollow.

The maximal hollow 3-polytopes (d'après AKW2016)



Hollow projections of hollow polytopes

Finiteness of the number of hollow 3-polytopes that *do not project* to lower dimensions is a general fact:

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Examples

P projects to a hollow 1-polytope $\Leftrightarrow$ P has width one.
P projects to a hollow 2-polytope $\Leftrightarrow$ P either has width one or projects to the second dilation of a unimodular triangle.

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$4 \neq 5$: In dimension ≥ 5 there are non-cyclic empty simplices.

The complete classification of empty 4-simplices (Iglesias-S., 2018+)

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There are 2461 of them. Their determinants range from 24 to 419. There is one of width 4 (determinant=101), 178 of width three (dets.$\in [49, 179]$), and the rest have width two (as predicted by Haase-Ziegler).

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We prove this in two parts:

1. The case of width at least three.
2. The case of width two.

Idea of proof for width ≥ 3

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Q is not hollow, but still has width ≥ 3.

We call *rational diameter* $\delta(P)$ of P the maximum length (w.r.t. the lattice) of a rational segment contained in P. It equals $\lambda_1^{-1}(P - P)$, where $\lambda_1(C) \equiv$ first successive minimum of C.

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If P is a simplex this can be improved to $\text{Vol}(P) \leq \frac{2^d d!}{\binom{2d}{d}}\delta(P)^d$

Bounding Vol(P) from Vol(Q)

Lemma

Let $\pi : P \to Q$ be an integer projection of a **hollow** *d-simplex P onto a* **non-hollow** *$(d-1)$-polytope Q. Let:*

- *$x \in Q$ be the Radon point of the projection.*
- *δ be the length of $\pi^{-1}(x)$.*
- *$0 < r < 1$ be the maximum dilation factor such that $Q_r := x + r(Q - x)$ is hollow.*

Then:

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- In what follows we project along the direction with δ=diameter(P).
- r measures whether Q is "close to hollow" ($r \simeq 1$) or "far from hollow" ($r \simeq 0$)

An upper bound for the volume of empty 4-simplices

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  E.g., whenever $r \leq 0.81$ we have $\delta \leq 1/0.19$ and

$$\mathrm{Vol}(P) \leq \frac{5.48}{0.19^4} = 4210.$$

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An upper bound for the volume of empty 4-simplices

Now, suppose that $\pi : P \to Q$ is the projection along the direction giving the rational diameter of P, so that the δ in the theorem equals the rational diameter of P. We have a dichotomy:

- If Q is "close to hollow" then we use the Lemma:

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Putting this together we get "Theorem 2":

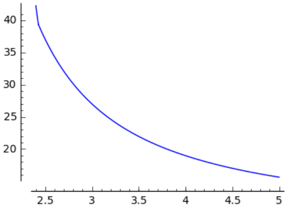
$$\mathrm{Vol}(P) \leq \frac{\delta}{r^3}\,\mathrm{Vol}(Q_r) \leq \cdots \leq 7600$$

A bound on the volume of wide 3-polytopes

Lemma (Iglesias-S. 2017+, inspired in AKW 2016)

Let K be a hollow convex 3-body of width $w > 1 + \frac{2}{\sqrt{3}} = 2.155$. Then,

$$\mathrm{vol}(K) \le \begin{cases} 8w^3/(w-1)^3, & \text{if } w \ge \frac{2}{\sqrt{3}}(\sqrt{5}-1)+1 = 2.427, \\ 3w^3/4(w-(1+2/\sqrt{3})), & \text{if } w \le 2.427. \end{cases}$$



Idea of proof for width $= 2$

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W.l.o.g. suppose $P \subset [-1, 1] \times \mathbb{R}^3$, and let $Q = P \cap (\{0\} \times \mathbb{R}^3)$. Then, by Schwarz symmetrization:

$$\mathrm{Vol}(P) \leq 2^4 \, \mathrm{Vol}(Q).$$

Hence, it suffices to show that $\mathrm{Vol}(Q) \leq 7600/16 = 475$.

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Hence, it suffices to show that $\mathrm{Vol}(Q) \leq 7600/16 = 475$.
Observe Q is half-integer. Two cases:

1. $\mathrm{width}(Q) \geq 5/2 \Rightarrow$ since Q is hollow,

$$\mathrm{Vol}(Q) = 6\,\mathrm{vol}\,Q \leq 6\frac{8(5/2)^3}{(3/2)^3} = 222.2$$

Idea of proof for width = 2

Let P be a hollow lattice 4-simplex of width = 2 that *does not project to a hollow 3-polytope*.

W.l.o.g. suppose $P \subset [-1,1] \times \mathbb{R}^3$, and let $Q = P \cap (\{0\} \times \mathbb{R}^3)$. Then, by Schwarz symmetrization:

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2. $\mathrm{width}(Q) \leq 2 \Rightarrow$ we apply to the middle slice of Q (call it R) the same ideas: R is a lattice-free polygon which does not project to dimension 1 $\Rightarrow$ (we skip details...) $\mathrm{Vol}(Q) \leq 324$

Empty 4-simplices of prime volume

Motivated by their equivalence to *terminal quotient singularities*, Mori, Morrison and Morrison (1989) studied empty 4-simplices of *prime determinant* and found that:

1. There are 1+1+29 infinite families with three, two, and one parameters respectively.
2. Up to determinant 419 there are some 4-simplices not in those families, but between 420 and 1600 there are none.

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> CONJECTURE 1.4 (four-dimensional terminal lemma). *Fix* $p \geq 421$. *Up to the actions of* $(\mathbf{Z}/p\mathbf{Z})^*$ *and* $\mathbf{S}^4$, *each isolated four-dimensional terminal* $\mathbf{Z}/p\mathbf{Z}$*-quotient singularity of index* p *is associated with one of the* p*-terminal quintuples given in Theorem* 1.3.

This conjecture was proved (modulo the "finitely many exceptions") by Bover (2009) (partially by Sankaran 1990)

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This conjecture was proved (modulo the "finitely many exceptions") by Bover (2009) (partially by Sankaran 1990) $\Rightarrow$ Complete classification of empty simplices of prime volume.

Empty 4-simplices of prime volume

THEOREM 1.3. *Let Q be a quintuple of integers summing to zero, and let p be a prime number. Suppose that either*

(a) $Q = (\alpha, -\alpha, \beta, \gamma, -\beta - \gamma)$ *with* $0 < |\alpha|, |\beta|, |\gamma| < p/2$, *and* $\beta + \gamma \neq 0$, *or*

(b) $Q = (\alpha, -2\alpha, \beta, -2\beta, \alpha + \beta)$ *with* $0 < |\alpha|, |\beta| < p/2$, *and* $\alpha + \beta \neq 0$, *or*

(c) *Q is one of the 29 quintuples listed in Table 1.9 and* $p > M_Q$.

Then Q is p-terminal.

TABLE 1.9

Stable Quintuple	*Linear Relations*
(9, 1, −2, −3, −5)	02100, 11002, 20122
(9, 2, −1, −4, −6)	01200, 02010, 20212
(12, 3, −4, −5, −6)	02001, 10002, 12220
(12, 2, −3, −4, −7)	02010, 11002, 20212
(9, 4, −2, −3, −8)	01200, 02001, 20221
(12, 1, −2, −3, −8)	02100, 12021, 20122
(12, 3, −1, −6, −8)	02010, 10020, 12202
(15, 4, −5, −6, −8)	02001, 20221
(12, 2, −1, −4, −9)	01200, 02010, 20212
(10, 6, −2, −5, −9)	02120, 10020, 12202
(15, 1, −2, −5, −9)	02100, 20122
(12, 5, −3, −4, −10)	02001, 02210, 20221
(15, 2, −3, −4, −10)	02010, 20212
(6, 4, 3, −1, −12)	02221, 20001
(7, 5, 3, −1, −14)	02221, 20001
(9, 7, 1, −3, −14)	02001, 20221
(15, 7, −3, −5, −14)	02001, 20221
(8, 5, 3, −1, −15)	02211, 20011
(10, 6, 1, −2, −15)	00210, 22012
(12, 5, 2, −4, −15)	00210, 22012
(9, 6, 4, −1, −18)	02221, 20001
(9, 6, 5, −2, −18)	02221, 20001
(12, 9, 1, −4, −18)	02001, 20221
(10, 7, 4, −1, −20)	02221, 20001
(10, 8, 3, −1, −20)	02221, 20001
(10, 9, 4, −3, −20)	02221, 20001
(12, 10, 1, −3, −20)	02001, 20221
(12, 8, 5, −1, −24)	02221, 20001
(15, 10, 6, −1, −30)	02221, 20001

Theorem 3

(Almost) Theorem 3 (Barile, Bernardi, Borisov and Kantor, 2011)

All but finitely many empty 4-simplices belong to the $29 + 1 + 1$ families of Mori-Morrison-Morrison (1988), all of which have width one or two.

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This is only true for 4-simplices of *prime* volume.

The correct version is:

Theorem 3 (Iglesias, Santos, 2018+)

All empty 4-simplices that project to hollow 3-polytopes belong to:

1. The 3-parameter family with quintuple $(a, -a, b, c, -b - c)$.
2. One of the two 2-parameter families with quintuples $(a, -2a, b, -2b, a + b)$ and $(a, -2a, b, -2b, a + b)$.
3. One of the $29 + 23$ one-parameter families given by the 29 quintuples of Mori, Morrison and Morrison (1988) or the new 23 non-primitive quintuples.

Cyclic simplices represented as $(d+1)$-tuples

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What are these "quintuples"

For each choice of $D \in \mathbb{N}$, a quintuple $v = (v_0, v_1, v_2, v_3, v_4)$ represents "the" cyclic simplex Δ in which v/D are the barycentric coordinates for a generator of $\mathbb{Z}^4/\Lambda(D)$.

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Remarks:

- All empty 4-simplces are cyclic (Barile et al 2011), so they can be represented in this way.

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- D equals the determinant of Δ.
- the v_i's are integers, and they are important only modulo D.
- if we choose $\sum v_i = 0$ and do not specify D, then a quintuple $(v_0, v_1, v_2, v_3, v_4)$ represents an infinite family of simplices, one for each D.

Interpretation of the quintuples

Each quintuple is a 1-parameter family of empty 4-simplices that project to a particular hollow 3-polytope.

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- Divided by D, they are barycentric coordinates for a generator of the (cyclic) group $\mathbb{Z}^4/L(\Delta)$.
- They are homogeneous coordinates for a line $\ell \in \{x \in \mathbb{R}^5 : \sum x_i = 1\} \cong \mathbb{R}^4$ passing through the origin (assumed to be a vertex of Δ). This line gives the projection direction, and has the property that the projection of Δ is hollow.

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- It gives the (unique) affine dependence among the projection of the vertices of Δ in the direction of the line ℓ.

Interpretation of the quintuples

More generally: a k-parameter family corresponds to the set of all d-dimensional lifts of a certain configuration of $d+1$ points in dimension $d-k$. The "k-parameter $(d+1)$-tuple" parametrizes the affine dependences among the $d+1$ points in $\mathbb{R}^k$.

In particular, the Nill-Ziegler result ("all except finitely many hollow d-polytopes project to a hollow $< d$-polytope") implies:

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In particular, the Nill-Ziegler result ("all except finitely many hollow d-polytopes project to a hollow $< d$-polytope") implies:

Corollary

In any fixed dimension d, the set of all hollow d-simplices can be stratified "à la Mori et al." into a finite number of "families". Each family is represented as a k-dimensional rational linear subspace of $\mathbb{R}^{d+1}$ ($k \in \{0, \ldots, d-1\}$). A k-parameter family corresponds to simplices projecting to a particular configuration A of $d+1$ points in $\mathbb{R}^k$ such that $\text{conv}(A)$ *is hollow but does not project to dimension $< d-k$.*

Proof of Theorem 3

The list in the statement corresponds to empty 4-simplices projectiong to lower dimensional hollow polytopes:

- Simplices projecting to dim 1 (that is, of width one) can a priori project in two ways: "$4+1$" or "$3+2$". But the classification of 3-dimensional empty simplices implies that the former is a special case of the latter. Affine dependences in the latter are parametrized by $(a, -a, b, c, -b-c)$ (the 3-parameter family of MMM).

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- A lattice 4-simplex Δ projecting to dim 2 must project to the second dilation of a unimodular triangle. For Δ to be empty one needs the vertices to project to one of the following configurations:

projection:

aff. dependence: $(a, -2a, b, -2b, a+b)$ $(a, -2a, b, -2b, a+b)$

Proof of Theorem 3 (cont.)

- Lattice 4-simplices projecting to dim. 3 can be exhaustively described via the (finite) classification of hollow 3-polytopes with at most 5 vertices and not projecting to dim two (Averkov et al. 2016).

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  To narrow the search we use that, of the three types of 3-polytopes with ≤ 5 vertices (tetrahedron, sq. pyramid, triang. bipyramid) only the latter can possibly produce infinitely many hollow 4-dimensional lifts (Blanco-Haase-Hofmann-S. 2016).

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  In this way we recover the 29 quintuples of Mori-Morrison-Morrison 1988, plus 23 additional "non-primitive quintuples".

The 29 quintuples

$\mathbb{Q}\{(9,1,-2,-3,-5)\}$
$\mathbb{Q}\{(9,2,-1,-4,-6)\}$
$\mathbb{Q}\{(12,3,-4,-5,-6)\}$
$\mathbb{Q}\{(12,2,-3,-4,-7)\}$
$\mathbb{Q}\{(9,4,-2,-3,-8)\}$
$\mathbb{Q}\{(12,1,-2,-3,-8)\}$
$\mathbb{Q}\{(12,3,-1,-6,-8)\}$
$\mathbb{Q}\{(15,4,-5,-6,-8)\}$
$\mathbb{Q}\{(12,2,-1,-4,-9)\}$
$\mathbb{Q}\{(10,6,-2,-5,-9)\}$
$\mathbb{Q}\{(15,1,-2,-5,-9)\}$
$\mathbb{Q}\{(12,5,-3,-4,-10)\}$
$\mathbb{Q}\{(15,2,-3,-4,-10)\}$
$\mathbb{Q}\{(6,4,3,-1,-12)\}$
$\mathbb{Q}\{(7,5,3,-1,-14)\}$
$\mathbb{Q}\{(9,7,1,-3,-14)\}$
$\mathbb{Q}\{(15,7,-3,-5,-14)\}$
$\mathbb{Q}\{(8,5,3,-1,-15)\}$
$\mathbb{Q}\{(10,6,1,-2,-15)\}$
$\mathbb{Q}\{(12,5,2,-4,-15)\}$
$\mathbb{Q}\{(9,6,4,-1,-18)\}$
$\mathbb{Q}\{(9,6,5,-2,-18)\}$
$\mathbb{Q}\{(12,9,1,-4,-18)\}$
$\mathbb{Q}\{(10,7,4,-1,-20)\}$
$\mathbb{Q}\{(10,8,3,-1,-20)\}$
$\mathbb{Q}\{(10,9,4,-3,-20)\}$
$\mathbb{Q}\{(12,10,1,-3,-20)\}$
$\mathbb{Q}\{(12,8,5,-1,-24)\}$
$\mathbb{Q}\{(15,10,6,-1,-30)\}$

The 29 quintuples of Mori-Morrison-Morrison. Each represents (the rational points in) a line through the origin, in the 4-torus $\mathbb{R}^4/L(\Delta)$.

The 23 "non-primitive quintuples"

$(0, 0, \frac{1}{2}, \frac{1}{2}, 0)$	+	$\mathbb{Q}\{(6, -2, -12, 4, 4)\}$	$(0, 0, \frac{2}{3}, \frac{1}{3}, 0)$	+	$\mathbb{Q}\{(-9, 6, 3, 3, -3)\}$
$(\frac{1}{2}, 0, 0, 0, \frac{1}{2})$	+	$\mathbb{Q}\{(8, -6, 2, -8, 4)\}$	$(\frac{1}{3}, 0, \frac{2}{3}, 0, 0)$	+	$\mathbb{Q}\{(9, -9, 3, -6, 3)\}$
$(0, 0, \frac{1}{2}, 0, \frac{1}{2})$	+	$\mathbb{Q}\{(8, -4, -12, 6, 2)\}$	$(0, 0, \frac{1}{3}, \frac{2}{3}, 0)$	+	$\mathbb{Q}\{(-9, 3, 6, 6, -6)\}$
$(\frac{1}{2}, 0, 0, 0, \frac{1}{2})$	+	$\mathbb{Q}\{(4, 6, -2, -16, 8)\}$	$(0, 0, \frac{1}{3}, \frac{2}{3}, 0)$	+	$\mathbb{Q}\{(12, -6, -12, 3, 3)\}$
$(0, \frac{1}{2}, \frac{1}{2}, 0, 0)$	+	$\mathbb{Q}\{(2, -12, 4, 12, -6)\}$	$(\frac{1}{3}, 0, \frac{2}{3}, 0, 0)$	+	$\mathbb{Q}\{(9, -18, 6, 6, -3)\}$
$(\frac{1}{2}, 0, \frac{1}{2}, 0, 0)$	+	$\mathbb{Q}\{(12, -16, 8, -6, 2)\}$	$(\frac{1}{3}, 0, \frac{2}{3}, 0, 0)$	+	$\mathbb{Q}\{(12, -18, 3, 6, -3)\}$
$(0, \frac{1}{2}, 0, 0, \frac{1}{2})$	+	$\mathbb{Q}\{(2, 12, -8, -12, 6)\}$	$(\frac{1}{3}, 0, \frac{2}{3}, 0, 0)$	+	$\mathbb{Q}\{(12, -9, 3, -12, 6)\}$
$(\frac{1}{2}, 0, 0, 0, \frac{1}{2})$	+	$\mathbb{Q}\{(8, 6, -2, -24, 12)\}$	$(\frac{1}{3}, 0, \frac{2}{3}, 0, 0)$	+	$\mathbb{Q}\{(6, -3, 6, -18, 9)\}$
$(0, \frac{1}{2}, 0, 0, \frac{1}{2})$	+	$\mathbb{Q}\{(6, -2, 8, -24, 12)\}$	$(0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	+	$\mathbb{Q}\{(3, -18, 6, 18, -9)\}$
$(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0, 0)$	+	$\mathbb{Q}\{(12, -12, 4, -8, 4)\}$	$(\frac{1}{6}, 0, 0, \frac{2}{3}, \frac{1}{6})$	+	$\mathbb{Q}\{(6, -18, 6, 12, -6)\}$
$(0, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2})$	+	$\mathbb{Q}\{(4, 8, -4, -16, 8)\}$			
$(0, 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4})$	+	$\mathbb{Q}\{(4, -16, 4, 16, -8)\}$			
$(0\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2})$	+	$\mathbb{Q}\{(4, 12, -4, -24, 12)\}$			

The 23 non-primitive quintuples. Each represents (the rational points in) a line in $\mathbb{R}^4/\Lambda(\Delta)$ not passing through the origin.

Theorem 2 (enumeration)

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With determinant ≤ 7600 there are 2461 empty 4-simplices that do not project to hollow 3-polytopes. Their determinants range from 24 to 419.

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The proof is via an exhaustive computer enumeration.

Note: It is easy to prove (by induction on the dimension) that there are finitely many lattice polytopes of a given dimension d and with normalized volume bounded by D, for every $d, D \in \mathbb{N}$ (e.g., Lagarias-Ziegler, 1991).

The algorithm implicit in the general proof is impracticable, but for the case of simplices another methods can be used.

Enumeration algorithms

To enumerate all empty 4-simplices of a given volume D we use one of two algorithms:

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- Algorithm 1: **If D has less than 5 prime factors**, then every empty 4-simplex Δ of volume D has a unimodular facet (because Δ is cyclic, by Barile et al. 2011, which implies the volumes of facets are relatively prime). Thus, Δ is equivalent to

  $$\mathrm{conv}\{e_1, e_2, e_3, e_4, v\},$$

  for some $v = (v_1, v_2, v_3, v_4) \in \mathbb{Z}^4$ with $\sum v_i = D + 1$. Moreover, v needs only to be considered modulo D, which gives a priori D^3 possibilities.

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- Algorithm 1: **If D has less than 5 prime factors**, then every empty 4-simplex Δ of volume D has a unimodular facet (because Δ is cyclic, by Barile et al. 2011, which implies the volumes of facets are relatively prime). Thus, Δ is equivalent to

$$\text{conv}\{e_1, e_2, e_3, e_4, v\},$$

  for some $v = (v_1, v_2, v_3, v_4) \in \mathbb{Z}^4$ with $\sum v_i = D + 1$. Moreover, v needs only to be considered modulo D, which gives a priori D^3 possibilities.

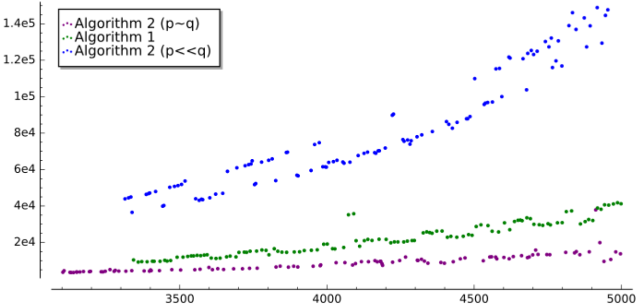
- Algorithm 2: **If D has at least 2 prime factors**, then we can decompose $D = pq$ with p and q relatively prime. Every 4-simplex Δ_D of volume D can be obtained by "merging" simplices Δ_p and Δ_q of volumes p and q.

Computational performance data

More than 10000 hours of computation have been used.

Computational performance data

More than 10000 hours of computation have been used.
Algorithm 2 is much slower than Algorithm 1 if $p << q$, and slightly faster than Algorithm 1 if $p \simeq q$.



Computation time (seconds) for the list of all empty lattice 4-simplices of a given volume

The "finitely many exceptions"

The enumeration gives us the 2461 empty 4-simplices that do not belong to the infinite families of Theorem 3. Their determinants range from 24 to 419.

Those of width ≥ 3 coincide with the list computed by Haase and Ziegler (2000): there are 178 of width three (with determinants in $[49, 179]$ and exactly one of width 4 (with determinant 101 and quintuple $(-1, 6, 14, 17, 65)$).

Nbr. of sporadic 4-simplices (part 1 of 2)

$V=24$:	1	$V=53$:	38	$V=78$:	3	$V=103$:	51	$V=129$:	17
$V=27$:	1	$V=54$:	11	$V=79$:	55	$V=104$:	8	$V=130$:	2
$V=29$:	3	$V=55$:	20	$V=80$:	7	$V=105$:	7	$V=131$:	29
$V=30$:	2	$V=56$:	3	$V=81$:	18	$V=106$:	8	$V=132$:	5
$V=31$:	2	$V=57$:	16	$V=82$:	13	$V=107$:	54	$V=133$:	14
$V=32$:	3	$V=58$:	13	$V=83$:	60	$V=108$:	5	$V=134$:	8
$V=33$:	4	$V=59$:	51	$V=84$:	7	$V=109$:	44	$V=135$:	6
$V=34$:	5	$V=60$:	4	$V=85$:	27	$V=110$:	5	$V=136$:	6
$V=35$:	3	$V=61$:	38	$V=86$:	11	$V=111$:	13	$V=137$:	28
$V=37$:	6	$V=62$:	26	$V=87$:	24	$V=112$:	2	$V=138$:	2
$V=38$:	8	$V=63$:	17	$V=88$:	5	$V=113$:	40	$V=139$:	37
$V=39$:	9	$V=64$:	9	$V=89$:	55	$V=114$:	4	$V=140$:	5
$V=40$:	1	$V=65$:	27	$V=90$:	6	$V=115$:	21	$V=141$:	6
$V=41$:	14	$V=66$:	3	$V=91$:	18	$V=116$:	11	$V=142$:	9
$V=42$:	5	$V=67$:	41	$V=92$:	9	$V=117$:	10	$V=143$:	13
$V=43$:	20	$V=68$:	13	$V=93$:	17	$V=118$:	9	$V=144$:	1
$V=44$:	8	$V=69$:	26	$V=94$:	12	$V=119$:	22	$V=145$:	14
$V=45$:	6	$V=70$:	4	$V=95$:	35	$V=120$:	3	$V=146$:	5
$V=46$:	7	$V=71$:	50	$V=96$:	3	$V=121$:	18	$V=147$:	10
$V=47$:	30	$V=72$:	3	$V=97$:	46	$V=122$:	9	$V=148$:	7
$V=48$:	5	$V=73$:	44	$V=98$:	9	$V=123$:	17	$V=149$:	26
$V=49$:	17	$V=74$:	18	$V=99$:	13	$V=124$:	8	$V=150$:	2
$V=50$:	8	$V=75$:	22	$V=100$:	8	$V=125$:	25	$V=151$:	19
$V=51$:	16	$V=76$:	14	$V=101$:	41	$V=127$:	24	$V=152$:	6
$V=52$:	6	$V=77$:	19	$V=102$:	3	$V=128$:	9	$V=153$:	9

Nbr. of sporadic 4-simplices (part 2 of 2)

$V = 154$:	3	$V = 181$:	13	$V = 211$:	4	$V = 245$:	3	$V = 293$:	5
$V = 155$:	12	$V = 182$:	5	$V = 212$:	2	$V = 247$:	3	$V = 299$:	2
$V = 156$:	2	$V = 183$:	5	$V = 213$:	3	$V = 248$:	3	$V = 304$:	1
$V = 157$:	11	$V = 184$:	5	$V = 214$:	2	$V = 249$:	2	$V = 308$:	1
$V = 158$:	10	$V = 185$:	7	$V = 215$:	5	$V = 250$:	1	$V = 310$:	1
$V = 159$:	9	$V = 186$:	2	$V = 216$:	1	$V = 251$:	5	$V = 311$:	1
$V = 160$:	3	$V = 187$:	7	$V = 218$:	5	$V = 254$:	1	$V = 313$:	1
$V = 161$:	13	$V = 188$:	5	$V = 219$:	4	$V = 256$:	2	$V = 314$:	1
$V = 163$:	17	$V = 189$:	2	$V = 220$:	1	$V = 257$:	3	$V = 317$:	1
$V = 164$:	6	$V = 190$:	2	$V = 221$:	3	$V = 259$:	2	$V = 319$:	2
$V = 165$:	1	$V = 191$:	8	$V = 222$:	1	$V = 261$:	1	$V = 321$:	1
$V = 166$:	7	$V = 192$:	1	$V = 223$:	7	$V = 263$:	7	$V = 323$:	1
$V = 167$:	18	$V = 193$:	12	$V = 225$:	2	$V = 265$:	1	$V = 331$:	1
$V = 168$:	3	$V = 194$:	3	$V = 226$:	4	$V = 267$:	1	$V = 332$:	1
$V = 169$:	13	$V = 196$:	4	$V = 227$:	9	$V = 268$:	1	$V = 334$:	2
$V = 170$:	2	$V = 197$:	13	$V = 229$:	6	$V = 269$:	2	$V = 335$:	1
$V = 171$:	6	$V = 199$:	11	$V = 230$:	3	$V = 271$:	4	$V = 347$:	1
$V = 172$:	3	$V = 200$:	4	$V = 232$:	1	$V = 272$:	1	$V = 349$:	2
$V = 173$:	15	$V = 201$:	3	$V = 233$:	9	$V = 274$:	1	$V = 353$:	1
$V = 174$:	3	$V = 202$:	2	$V = 234$:	1	$V = 275$:	1	$V = 355$:	1
$V = 175$:	8	$V = 203$:	7	$V = 235$:	3	$V = 278$:	2	$V = 356$:	1
$V = 176$:	4	$V = 204$:	1	$V = 237$:	1	$V = 283$:	2	$V = 376$:	1
$V = 177$:	5	$V = 205$:	4	$V = 238$:	2	$V = 287$:	1	$V = 377$:	2
$V = 178$:	2	$V = 206$:	4	$V = 239$:	3	$V = 289$:	4	$V = 397$:	1
$V = 179$:	21	$V = 207$:	2	$V = 241$:	6	$V = 290$:	1	$V = 398$:	1
$V = 180$:	1	$V = 208$:	1	$V = 244$:	2	$V = 291$:	1	$V = 419$:	1
		$V = 209$:	10			$V = 292$:	1		

Nbr. of sporadic t.q.s. of prime volume (MMM vs. us)

TABLE 1.14

p	S_p	p	S_p	p	S_p	p	S_p
2	0	73	220	179	105	283	10
3	0	79	275	181	65	293	25
5	0	83	300	191	40	307	0
7	0	89	275	193	60	311	5
11	0	97	230	197	65	313	5
13	0	101	201	199	55	317	5
17	9	103	255	211	20	331	5
19	13	107	270	223	35	337	0
23	28	109	220	227	45	347	5
29	39	113	200	229	30	349	10
31	30	127	120	233	45	353	5
37	50	131	145	239	15	359	0
41	76	137	140	241	30	367	0
43	110	139	185	251	25	373	0
47	100	149	130	257	15	379	0
53	195	151	95	263	35	383	0
59	260	157	55	269	10	389	0
61	186	163	85	271	20	397	5
67	205	167	90	277	0	409	0
71	250	173	75	281	0	419	5

$V = 29$:	15	$V = 113$:	200	$V = 229$:	30
$V = 31$:	10	$V = 127$:	120	$V = 233$:	45
$V = 37$:	30	$V = 131$:	145	$V = 239$:	15
$V = 41$:	66	$V = 137$:	140	$V = 241$:	30
$V = 43$:	100	$V = 139$:	185	$V = 251$:	25
$V = 47$:	150	$V = 149$:	130	$V = 257$:	15
$V = 53$:	190	$V = 151$:	95	$V = 263$:	35
$V = 59$:	255	$V = 157$:	55	$V = 269$:	10
$V = 61$:	186	$V = 163$:	85	$V = 271$:	20
$V = 67$:	205	$V = 167$:	90	$V = 283$:	10
$V = 71$:	250	$V = 173$:	75	$V = 293$:	25
$V = 73$:	220	$V = 179$:	105	$V = 311$:	5
$V = 79$:	275	$V = 181$:	65	$V = 313$:	5
$V = 83$:	300	$V = 191$:	40	$V = 317$:	5
$V = 89$:	275	$V = 193$:	60	$V = 331$:	5
$V = 97$:	230	$V = 197$:	65	$V = 347$:	5
$V = 101$:	201	$V = 199$:	55	$V = 349$:	10
$V = 103$:	255	$V = 211$:	20	$V = 353$:	5
$V = 107$:	270	$V = 223$:	35	$V = 397$:	5
$V = 109$:	220	$V = 227$:	45	$V = 419$:	5

Thank you for your attention

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