Isoperimetric inequalities in convex sets (Part II)

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Properties of the isoperimetric profile I_C

Summary of part I

Remember we have proven so far

- Existence of isoperimetric regions in C
- Continuity of I_C
- Symmetry of I_C (w.r.t. $|C|/2$)
- Continuous extension to $v = 0, |C|$
- Positivity of I_C

Properties of the isoperimetric profile I_C

Define the normalized isoperimetric profile $J_C : (0,1) \to \mathbb{R}^+$ by $J_C(\lambda) := I_C(\lambda|C|)$

Theorem

Let $(C_i)_{i\in\mathbb{N}}$ be a sequence of convex bodies converging in Hausdorff distance to a convex body C. Then $(J_{C_i})_i$ converges pointwise to J_C.

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Proof

Similar to the continuity of the isoperimetric profile.
Fix some $\lambda \in (0,1)$, and take isoperimetric regions $E_i \subset C_i$ of volumes $\lambda|C_i|$. As the perimeters of the sets E_i are uniformly bounded, we can extract a subsequence converging in L^1 topology to some set $E \subset C$ of volume $\lambda|C|$. We have

$$J_C(\lambda) \leq P_C(E) \leq \liminf_{i\to\infty} P_{C_i}(E_i) = \liminf_{i\to\infty} J_{C_i}(\lambda).$$

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Proof (continuation)

Now we apply the following construction: as $C_i \to C$ in Hausdorff distance, possibly after a translation so that 0 is an interior point for almost all i, we can take a sequence μ_i converging to 1 so that $C_i \subset \mu_i C$. Fix an isoperimetric region $E \subset C$ of volume $\lambda|C|$ and consider the sets $\mu_i E \subset \mu_i C$, we restrict them to C_i and we add or remove a small ball B_i so that we get a set $E_i^* \subset C_i$ with volume $\lambda|C_i|$. Then we have

$$\begin{aligned}\limsup_{i\to\infty} J_{C_i}(\lambda) \leq \limsup_{i\to\infty} P_{C_i}(E_i^*) &\leq \limsup_{i\to\infty} P_{C_i}(\mu_i E) + P(B_i) \\ &\leq \limsup_{i\to\infty} \left(P_{\mu_i C}(\mu_i E) + P(B_i)\right) \\ &= \limsup_{i\to\infty} \left(\mu_i^{n-1} P_C(E) + P(B_i)\right) \\ &= P_C(E) = J_C(\lambda) \quad \square\end{aligned}$$

Properties of the isoperimetric profile

We prove now the fundamental property

Theorem

Let $C \subset \mathbb{R}^n$ be a convex body with smooth boundary ($C^{2,\alpha}$ is enough). Then $I_C^{n/(n-1)}$ is a concave function.

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We shall make use of the following elementary Lemma

Lemma

Let $f : (a, b) \to \mathbb{R}$ be a continuous function. Assume that, for all $x \in (a, b)$, there is a smooth function f_x, defined on a neighborhood of x such that $f(z) \leq f_x(z)$, $f(x) = f_x(x)$, and $f_x''(x) \leq 0$. Then f is a concave function.

Properties of the isoperimetric profile

Proof (of Theorem)

We follow an argument by Kuwert.
Assume first that isoperimetric regions have smooth boundaries.
Fix $v_0 \in (0, |C|)$. Let E be an isoperimetric region of volume v_0, and S the closure of $\partial E \cap \text{int}(C)$. By hypothesis S is a smooth hypersurface meeting orthogonally the boundary of C. Take a vector field X in $\mathbb{R}^n$ with compact support so that it is tangent to ∂C, and so that $\langle X, N \rangle = 1$. Let $\{\varphi_t\}_{t \in \mathbb{R}}$ be the flow associated to X. Since

$$\frac{d}{dt}\bigg|_{t=0} |\varphi_t(E)| = \int_S \langle X, N \rangle = A(S) > 0,$$

we can describe the deformation using the volume as a parameter near v_0. Let $A(v)$ be the function describing the perimeter of $\varphi_{t(v)}(E)$ as a function of v.

Properties of the isoperimetric profile

Proof of Theorem (continuation)

The second variation formulas for the perimeter and the volume yield

$$\frac{d^2A^{n/(n-1)}}{dv^2} = \frac{1}{A^2}\frac{n}{n-1}\,A^{1/(n-1)}\bigg(-\int_S\bigg(|\sigma|^2-\frac{H^2}{n-1}\bigg)-\int_{\partial S}\mathrm{II}_{\partial C}(N,N)\bigg)$$

where $|\sigma|^2=\sum_{i=1}^{n-1}k_i^2$, H is the mean curvature of S, and $\mathrm{II}_{\partial C}$ is the second fundamental form of ∂C. The above quantity is less than or equal to 0 (since $\mathrm{II}_{\partial C}\geq 0$ by the convexity of C and $|\sigma|^2-H^2/(n-1)\geq 0$ by the arithmetic-geometric inequality).

Properties of the isoperimetric profile

Proof of Theorem (final)

In the general case (no regularity assumed), we need to use vector fields X_ε so that $\langle X_\varepsilon, N\rangle = \varphi_\varepsilon$, a family of functions approximating the function 1 in the Sobolev space of the regular part of ∂E. In the Lemma the function f_x must be replaced by a sequence of functions $(f_x)_i$ satisfying $\limsup_{i\to\infty}(f_x)_i''(x) \leq 0$. The proof runs without changes. □

Concavity of $I_C^{n/(n-1)}$ for general convex sets

This result was first proved by E. Milman

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Proof

It follows from approximation by convex sets with smooth boundary and the convergence of isoperimetric profiles in Hausdorff distance. □

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Remark

In particular, I_C is strictly concave, absolutely continuous, smooth almost everywhere and possesses left and right derivatives everywhere.

Consequences

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Let $C \subset \mathbb{R}^n$ be a convex body and $E \subset C$ an isoperimetric region. Then E and $C \setminus E$ are connected.

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Proof (Bayle)

Assume that E can be separated into two sets E_1, E_2 of volumes $v_1 + v_2 = v$. Then we have

$$I_C(v) = P_C(E) = P_C(E_1) + P_C(E_2) \geq I_C(v_1) + I_C(v_2).$$

But, since I_C is an strictly concave function with $I_C(0) = 0$, we have

$$I_C(v) < I_C(v_1) + I_C(v_2),$$

yielding a contradiction.
The second property follows since $C \setminus E$ is also an isoperimetric region.

Theorem

Let $C \subset \mathbb{R}^n$ be a convex body and $E \subset C$ an isoperimetric region. Assume that $\partial E \cap \text{int}(C)$ is smooth. Then $\partial E \cap \text{int}(C)$ is connected.

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Proof

We closely follow an argument by Samelson.
Let $S := \partial E \cap \text{int}(C)$. Take two points $p, q \in S$. As the boundary of E separates C, and E, $C \setminus E$ are connected, we can find a closed smooth curve meeting S only at p, q. We can find a smooth homothopy contracting this curve to a point and so we can find a C^∞ map of the disc meeting S transversally. By trivial topological arguments, p, q must be connected by a curve contained in S.

Thanks for your attention