Valuations on Convex Bodies

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Felix Klein's Erlangen Program 1872

Geometry is the study of invariants of transformation groups.

Groups acting on $\mathbb{R}^n$

- Group of rigid motions $\overline{\mathrm{SO}(n)}$: $x \mapsto Ux + b$
  where U is an orthogonal $n \times n$ matrix and $b \in \mathbb{R}^n$
- Special linear group $\mathrm{SL}(n)$: $x \mapsto Ax$
  where A is an $n \times n$ matrix of determinant 1
- General linear group $\mathrm{GL}(n)$: $x \mapsto Ax$
  where A is an $n \times n$ matrix of determinant $\neq 0$

Valuations on Convex Bodies

- $\mathcal{K}^n$ space of convex bodies (compact convex sets) in $\mathbb{R}^n$
- $\langle \mathbb{A}, + \rangle$ abelian semigroup
- $\mathrm{Z} : \mathcal{K}^n \to \mathbb{A}$ is a **valuation** $\iff$

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$.

Valuations on Convex Bodies

Theorem (Hadwiger 1952)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, rigid motion invariant valuation*

$$\Longleftrightarrow$$

$\exists \; c_0, c_1, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0 \, V_0(K) + \cdots + c_n \, V_n(K)$$

for every $K \in \mathcal{K}^n$.

Theorem (Blaschke 1937)

$\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists \, c_0, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(P) = c_0 \, V_0(P) + c_n \, V_n(P)$$

for every $P \in \mathcal{P}^n$.

Minkowski Valuations

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  Rolf Schneider (*Convex Bodies: The Brunn Minkowski Theory*, 1993; 2014)
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$$K + L \quad = \quad \{x + y : x \in K, y \in L\}$$

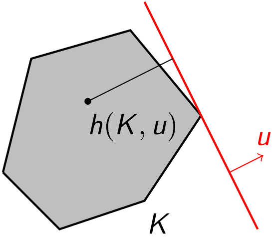
- $\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is a Minkowski valuation $\Leftrightarrow$

$$\mathrm{Z}\, K + \mathrm{Z}\, L = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

  for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$

Convex Bodies $\mathcal{K}^n$

- Support function $h(K,\cdot):\mathbb{R}^n \to \mathbb{R}$



  - $h(K,u) = \max\{u \cdot x : x \in K\}$
  - $h(K,u+v) \leqslant h(K,u) + h(K,v)$ sublinear

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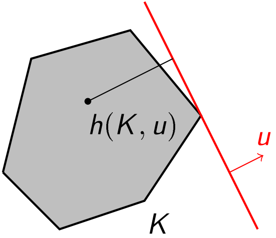

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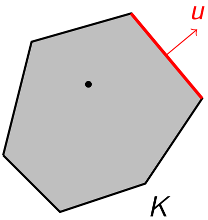
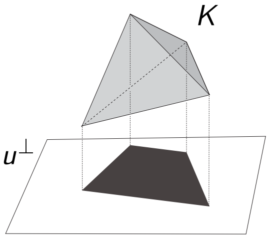
- Surface area measure $S(K,\cdot):\mathcal{B}(\mathbb{S}^{n-1}) \to [0,\infty)$



  - $S(K,\omega) = \mathcal{H}^{n-1}(\{x \in \mathrm{bd}\, K : u_K(x) \in \omega\})$
  - $S(K,\cdot)$ measure with centroid at origin, not concentrated on a great sphere

Integral Affine Surface Area

- Cauchy's surface area formula



$$S(K) = \frac{1}{v_{n-1}} \int_{\mathbb{S}^{n-1}} V_{n-1}(K|u^\perp)\, du$$

- $S(K)$ surface area of K
- $u^\perp$ hyperplane orthogonal to u
- $K|u^\perp$ projection of K to $u^\perp$
- v_k k-dimensional volume of k-dimensional unit ball

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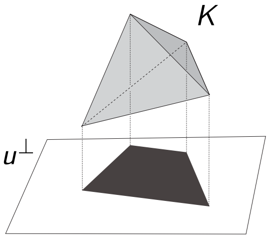

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$$I(K) = \Big(\frac{1}{n} \int_{\mathbb{S}^{n-1}} V_{n-1}(K|u^\perp)^{-n}\, du\Big)^{-1/n}$$

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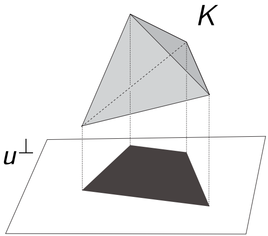

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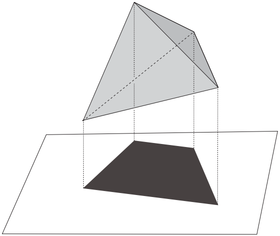
$$I(K) = \left(\frac{1}{n} \int_{\mathbb{S}^{n-1}} V_{n-1}(K|u^\perp)^{-n}\, du\right)^{-1/n}$$

  - $I(\phi K) = I(K)$ for all $\phi \in \mathrm{SL}(n)$

Projection Body

- Projection body, ΠK, of K (Minkowski 1901)



  - $h(\Pi K, u) = V_{n-1}(K|u^\perp)$
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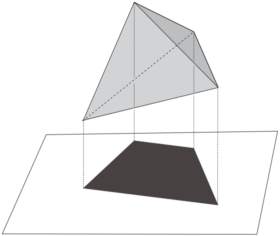

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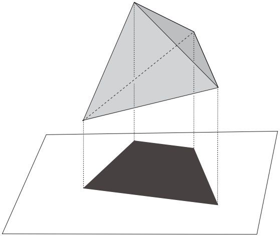

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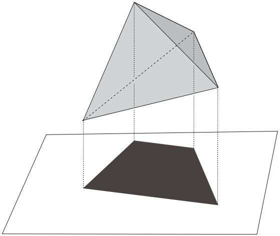

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- $\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is $\mathrm{GL}(n)$ contravariant of weight q $\Leftrightarrow$

$$\mathrm{Z}(\phi K) = |\det \phi|^q \, \phi^{-t} \, \mathrm{Z} \, K \qquad \forall \phi \in \mathrm{GL}(n)$$

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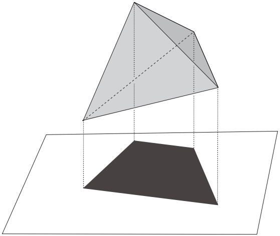

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- $V_n(\Pi^* K) = I(K)^{-n}$

Affine Isoperimetric Inequalities

Theorem (Petty: 1972)

$$V_n(K)^{n-1} V_n(\Pi^* K) \leqslant V_n(E)^{n-1} V_n(\Pi^* E)$$

for $K \in \mathcal{K}^n_{(0)}$, with equality if and only if K is an ellipsoid E

- $K^* = \{y \in \mathbb{R}^n : x \cdot y \leqslant 1 \text{ for all } x \in K\}$ polar body
- Petty's projection inequality is stronger than the Euclidean isoperimetric inequality.

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Theorem (Gaoyong Zhang: 1991)

$$V_n(K)^{n-1} V_n(\Pi^* K) \geqslant V(S)^{n-1} V(\Pi^* S)$$

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Classification of Minkowski Valuations

Theorem (L.: AiM 2002)

$\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ *is a continuous,* $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation*

$$\Longleftrightarrow$$

$\exists\, c \geqslant 0$:

$$\mathrm{Z}\, K = c\, \Pi\, K$$

for every $K \in \mathcal{K}^n$.

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- Ludwig (TAMS 2005, JDG 2010), Schuster (TAMS 2007, Duke 2010), Schuster & Wannerer (TAMS 2009, AJM 2015), Wannerer (IUMJ 2009), Haberl (AJM 2010, JEMS 2012), Abardia (JFA 2012, IMRN 2015), Abardia & Bernig (AiM 2011), Li, Yuan & Leng (TAMS 2015), Li & Leng (AiM 2015), ...

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Sketch of the Proof

- Homogeneous decomposition: $t \in \mathbb{Q}$, $t \geqslant 0$

$$h(\mathrm{Z}(t\,P), x) = \mathrm{Z}_0(P, x) + \mathrm{Z}_1(P, x)\, t + \cdots + \mathrm{Z}_n(P, x)\, t^n$$

with $\mathrm{Z}_i(\cdot, x)$ translation invariant, rational i-homogeneous valuation

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- $\mathrm{Z}_i(P,x) = 0$ for $i \leqslant j \;\; \Rightarrow$

$$\mathrm{Z}_{j+1}(P,x) = \lim_{t\to 0} \frac{h(\mathrm{Z}\,P, x)}{t^{j+1}}$$

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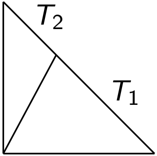
Sketch of the Proof

$\mathrm{Z} : \mathcal{P}^n \to \mathcal{K}^n$ SL(n) contravariant,
rational i-homogeneous,
translation invariant valuation



- $T_1 \cup T_2 = T$, $\dim(T_1 \cap T_2) < n$
- $\exists \phi_i \in \mathrm{GL}(n) : T_i = \phi_i T$

- $\mathrm{Z}\, T + \mathrm{Z}(T_1 \cap T_2) \quad = \quad \mathrm{Z}\, T_1 + \mathrm{Z}\, T_2$

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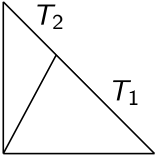

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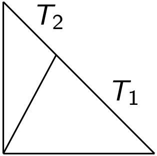

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- $f(x) = h(\mathrm{Z}\, T, x)$, $g(x) = h(\mathrm{Z}(T_1 \cap T_2), x)$

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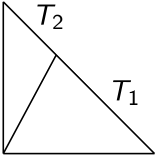

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- $f(x) = h(Z\, T, x)$, $g(x) = h(Z(T_1 \cap T_2), x)$
  $f(x) + g(x) = (1-t)^{\frac{i+1}{n}} f(\phi_1^{-1} x) + t^{\frac{i+1}{n}} f(\phi_2^{-1} x)$
  family of functional equations

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Theorem (Haberl: JEMS 2012)

$\mathrm{Z} : \mathcal{K}_0^n \to \mathcal{K}_0^n$ *is a continuous and* $\mathrm{SL}(n)$ *contravariant Minkowski valuations*

$$\Longleftrightarrow$$

$\exists c \geqslant 0$*:*

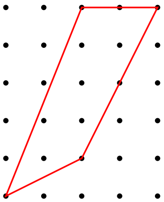
$$\mathrm{Z}\, K = c\, \Pi\, K$$

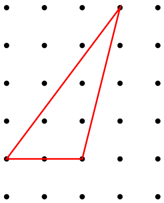
for every $K \in \mathcal{K}_0^n$.

- $\mathcal{K}_0^n$ convex bodies containing the origin
- Parapatits (TAMS 2014, JLMS 2014), Li & Leng (AiM 2015), ...

Lattice Polytopes

- P lattice polytope in $\mathbb{R}^n$
  $\iff P$ is the convex hull of finitely many points from $\mathbb{Z}^n$





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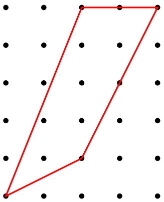

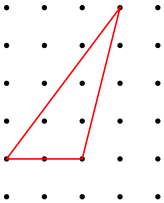

- Applications
  - Integer programming
  - Geometry of numbers
  - Algebraic geometry (Newton polytope)

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$$\Longleftrightarrow$$

$\mathrm{Z}(\phi P) = \phi^{-t}\, \mathrm{Z}(P)$ for all $\phi \in \mathrm{SL_n}(\mathbb{Z})$ and $P \in \mathcal{P}(\mathbb{Z}^n)$

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- $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{A}$ is translation invariant

$$\Longleftrightarrow$$

$\mathrm{Z}(P + x) = \mathrm{Z}(P)$ for all $x \in \mathbb{Z}^n$ and $P \in \mathcal{P}(\mathbb{Z}^n)$

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$\mathrm{Z}(\phi P) = \phi^{-t}\, \mathrm{Z}(P)$ for all $\phi \in \mathrm{SL_n}(\mathbb{Z})$ and $P \in \mathcal{P}(\mathbb{Z}^n)$

- $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{A}$ is translation invariant

$$\Longleftrightarrow$$

$\mathrm{Z}(P + x) = \mathrm{Z}(P)$ for all $x \in \mathbb{Z}^n$ and $P \in \mathcal{P}(\mathbb{Z}^n)$

Question.

Classification of valuations on $\mathcal{P}(\mathbb{Z}^n)$.

Classification of Minkowski Valuations

Theorem (L. & Böröczky: JEMS 2018)

$\operatorname{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{K}^n$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation*

$$\iff$$

$\exists\, c \geqslant 0$:

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Classification of Valuations on Lattice Polytopes

Theorem (Betke & Kneser: Crelle 1985)

$\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists \;\; c_0, c_1, \dots, c_n \in \mathbb{R}$:

$$\mathrm{Z}(P) = c_0 \, \mathrm{L}_0(P) + \cdots + c_n \, \mathrm{L}_n(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

- $\mathrm{L}_0(P), \dots, \mathrm{L}_n(P)$ coefficients of the Ehrhart polynomial

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- $\mathrm{L}_0(P), \dots, \mathrm{L}_n(P)$ coefficients of the Ehrhart polynomial
- $\mathrm{L}(P)$ number of points in $P \cap \mathbb{Z}^n$ for $P \in \mathcal{P}(\mathbb{Z}^n)$
- $$\mathrm{L}(kP) = \sum_{i=0}^{n} \mathrm{L}_i(P)\, k^i \quad \text{for } \; k \in \mathbb{N}$$

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Corollary (L. & Reitzner: DCG 2017)

$\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$ *is an* $\mathrm{SL}(n)$ *and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists \;\; c_0 \in \mathbb{R}$ *and Cauchy function* $\zeta : [0, \infty) \to \mathbb{R}$:

$$\mathrm{Z}(P) = c_0 \, V_0(P) + \zeta\big(V_n(P)\big)$$

for every $P \in \mathcal{P}_0^n$.

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Steps in the Proof

$$\mathrm{Z}\, P = c\, \Pi\, P$$

- Inclusion-exclusion principle (Betke 1979, McMullen 2009)
  Reduction to basic simplices (Betke & Kneser 1985)

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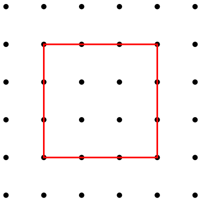

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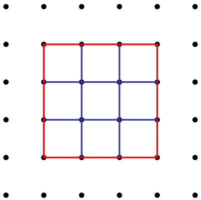

- $\mathrm{Z}(k\,[0,1]^n) + (k^n - k^{n-1}) \sum_{i=1}^{n} c\,[-e_i, e_i] = k^n\,\mathrm{Z}\,[0,1]^n$

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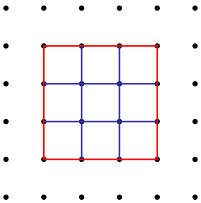

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- Homogeneous decomposition $\Rightarrow$ $\mathrm{Z}\,[0,1]^n = c\,\Pi\,[0,1]^n$

Steps in the Proof

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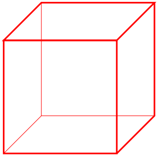

Steps in the Proof

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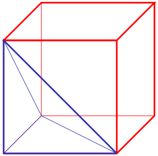

- $[0,1]^n = T_n \cup R_n \;\Rightarrow$

  $c\;\Pi\,[0,1]^n + c\,[-w,w] = \operatorname{Z} T_n + \operatorname{Z} R_n$

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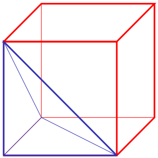

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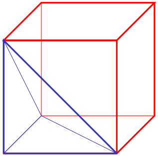

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- Symmetries $\Rightarrow \mathrm{Z}\,T_n = c\,\Pi\,T_n$

Classification of Minkowski Valuations

Theorem (L.: TAMS 2005)

$\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ *is a continuous,* $\mathrm{SL}(n)$ *covariant and translation invariant Minkowski valuation*

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$\exists\, c \geqslant 0$:

$$\mathrm{Z}\, K = c\, \mathrm{D}\, K$$

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for every $K \in \mathcal{K}^n$.

- $\mathrm{D}\, K = K + (-K)$ difference body of K

Classification of Minkowski Valuations

Theorem (Haberl: JEMS 2012)

For $n \geqslant 3$, the map $\mathrm{Z} : \mathcal{K}_0^n \to \mathcal{K}_0^n$ is a continuous, $\mathrm{SL}(n)$ equivariant Minkowski valuation

$$\Longleftrightarrow$$

$\exists\, c_1, \dots, c_4 \geqslant 0$:

$$\mathrm{Z}\, K = c_1\, K + c_2(-K) + c_3\, \mathrm{M}^+\, K + c_4\, \mathrm{M}^-\, K$$

for every $K \in \mathcal{K}_0^n$.

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- Asymmetric moment bodies:

  $h(\mathrm{M}^+ K, u) = \int_K (u \cdot x)_+ \, dx, \;\; h(\mathrm{M}^- K, u) = \int_K (u \cdot x)_- \, dx$

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- Classification of $\mathrm{GL}(n)$ equivariant Minkowski valuations (L.: TAMS 2005)
- Classification of L_p Minkowski valuations (Parapatits: JLMS 2014)

Classification of Minkowski Valuations

Theorem (Böröczky & L.: JEMS 2018)

$\operatorname{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{K}^n$ *is an* $\mathrm{SL_n}(\mathbb{Z})$ *equivariant and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists \;\; a, b \geqslant 0:$

$$\operatorname{Z} P = a(P - \ell_1(P)) + b(-P + \ell_1(P))$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

- $\ell_1(P)$ discrete Steiner point of P

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- $\ell_1(P)$ discrete Steiner point of P
- $\ell(P) = \sum_{x \in P \cap \mathbb{Z}^n} x$ discrete moment vector of P
- $$\ell(kP) = \sum_{i=1}^{n+1} \ell_i(P)\, k^i \quad \text{for } k \in \mathbb{N}$$

Classification of Minkowski Valuations

Theorem (L.: JDG 2010)

$\mathrm{Z} : \mathcal{K}^n_{(0)} \to \mathcal{K}^n$ *is a continuous,* $\mathrm{GL}(n)$ *contravariant of weight* q *and translation invariant Minkowski valuation*

$$\Longleftrightarrow$$

$\exists\, c_0 \in \mathbb{R}, c_1, c_2 \geqslant 0$:

$$\mathrm{Z}\,K = \begin{cases} c_1\,\Pi\,K & \text{for } q = 1 \\ c_1\,K^* + c_2(-K^*) & \text{for } q = 0 \\ c_0\,m(K^*) + c_1\,\mathrm{M}\,K^* & \text{for } q = -1 \\ \{0\} & \text{otherwise} \end{cases}$$

for every $K \in \mathcal{K}^n_{(0)}$.

- $m(K) = \int_K x\,dx$ moment vector
- $h(\mathrm{M}\,K, u) = \int_K |u \cdot x|\,dx$ moment body

Related Results

- **Classification of function valued valuations**

  Jin Li (IMRN 2018+, 2018+)

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- **Classification of translation invariant Minkowski valuations**

  Abardia, Colesanti, Saorín (AiM 2018)

Thank you!