# Valuations on Convex Bodies

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- Special affine group $\overline{\mathrm{SL}(n)}$: $x \mapsto Ax + b$
  where $A$ is an $n \times n$ matrix of determinant 1 and $b \in \mathbb{R}^n$
- General linear group $\mathrm{GL}(n)$: $x \mapsto Ax$
  where $A$ is an $n \times n$ matrix of determinant $\neq 0$

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- $\mathcal{K}^n$ space of convex bodies (compact convex sets) in $\mathbb{R}^n$
- $\langle \mathbb{A}, + \rangle$ abelian semigroup

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$$\operatorname{Z}(K) + \operatorname{Z}(L) = \operatorname{Z}(K \cup L) + \operatorname{Z}(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$.

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- Hilbert's Third Problem:
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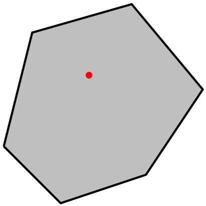
- Classification of valuations:

Blaschke 1937, **Hadwiger** 1949, Schneider 1971, Groemer 1972, McMullen 1977, Betke & Kneser 1985, Klain 1995, Ludwig 1999, Reitzner 1999, Alesker 1999, Hug 2005, Bernig 2006, Fu 2006, Haberl 2006, Schuster 2006, Tsang 2010, Wannerer 2010, Abardia 2011, Parapatits 2011, Faifman 2013, Solanes 2014, Wang 2014, Böröczky 2015, Li 2015, Ma 2016, Colesanti 2017, Mussnig 2017, ...

# Affine Classification Theorems

- $\mathcal{P}_0^n$ convex polytopes $P$ in $\mathbb{R}^n$ with $0 \in P$
- $\mathrm{Z} : \mathcal{P}_0^n \to \mathbb{R}$ *simple* $\Leftrightarrow \mathrm{Z}(P) = 0$ for $\dim(P) < n$



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- $\mathrm{Z} : \mathcal{P}_0^n \to \mathbb{R}$ simple and $\mathrm{SL}(n)$ invariant
  $\Rightarrow \exists\, \zeta : [0, \infty] \to \mathbb{R}$: $\mathrm{Z}(T) = \zeta(V_n(T))$
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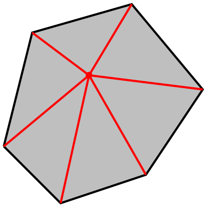

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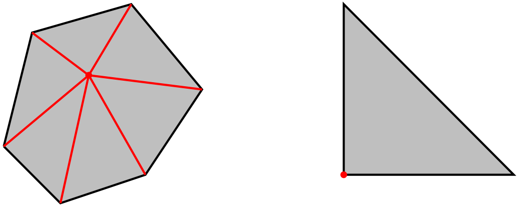

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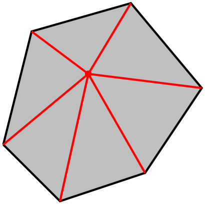

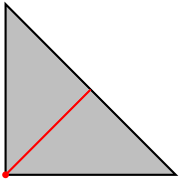

- $\mathrm{Z}(T) = \mathrm{Z}(T_1) + \mathrm{Z}(T_2) \Rightarrow \zeta(x+y) = \zeta(x) + \zeta(y)$ for $x, y \geqslant 0$
  $\Rightarrow \zeta$ solution of the Cauchy functional equation (Cauchy function)

# Affine Classification Theorems

**Theorem (L. & Reitzner: DCG 2017)**

$\mathrm{Z}: \mathcal{P}_0^n \to \mathbb{R}$ *is an* $\mathrm{SL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_0, c_0' \in \mathbb{R}$ *and Cauchy function* $\zeta: [0,\infty) \to \mathbb{R}$*:*

$$\mathrm{Z}(P) = c_0\, V_0(P) + c_0'\, (-1)^{\dim P}\, \mathbb{1}_{\operatorname{relint} P}(0) + \zeta\big(V_n(P)\big)$$

*for every* $P \in \mathcal{P}_0^n$.

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*for every* $P \in \mathcal{P}_0^n$.

**Corollary (Blaschke 1937)**

$\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_0, c_n \in \mathbb{R}$*:*

$$\mathrm{Z}(P) = c_0\, V_0(P) + c_n\, V_n(P)$$

*for every* $P \in \mathcal{P}^n$.

# Affine Classification Theorems

**Theorem (L.: Abh. Hamb. 1999; L. & Reitzner: AiM 1999)**

*$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is an upper semicontinuous, $\mathrm{SL}(n)$ and translation invariant valuation*

$$\Longleftrightarrow$$

*$\exists \;\; c_0, c_n \in \mathbb{R}, a \geqslant 0$:*

$$\mathrm{Z}(K) = c_0\, V_0(K) + c_n\, V_n(K) + a\,\Omega(K)$$

*for every $K \in \mathcal{K}^n$.*

- $\Omega(K) = \int_{\partial K} \kappa(K,x)^{\frac{1}{n+1}}\, dx$ affine surface area of $K$
- $\kappa(K,\cdot)$ Gaussian curvature

# GL$(n)$ and SL$(n)$ invariant Valuations on $\mathcal{K}^n_{(0)}$

- $\mathcal{K}^n_{(0)}$ convex bodies in $\mathbb{R}^n$ with $0 \in \operatorname{int} K$

# $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ invariant Valuations on $\mathcal{K}^n_{(0)}$

- $\mathcal{K}^n_{(0)}$ convex bodies in $\mathbb{R}^n$ with $0 \in \operatorname{int} K$
- $V_0(K)$ Euler characteristic

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- $V_n(K)$ volume

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- $V_0(K)$ Euler characteristic
- $V_n(K)$ volume
- $K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1 \text{ for } y \in K\}$ polar body of $K \in \mathcal{K}^n_{(0)}$
- $V_n(K^*)$ volume of $K^*$

$$(K \cup L)^* = K^* \cap L^*, \ \ (K \cap L)^* = K^* \cup L^*$$

for $K, L \in \mathcal{K}^n_{(0)}$ such that $K \cup L \in \mathcal{K}^n_{(0)}$

# Affine Classification Theorems

**Theorem (L. & Reitzner: Annals 2010)**

$\mathrm{Z} : \mathcal{K}^n_{(0)} \to \mathbb{R}$ *is an upper semicontinuous and* $\mathrm{GL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists \;\; c_0 \in \mathbb{R}, a \geqslant 0$*:*

$$\mathrm{Z}(K) = c_0\, V_0(K) + a\,\Omega_n(K)$$

*for every* $K \in \mathcal{K}^n_{(0)}$.

- $\Omega_n(K) = \displaystyle\int_{\partial K} \kappa_0(K,x)^{\frac{1}{2}}\, d\mu_K(x)$ centro-affine surface area of $K$
  - $\kappa_0(K,x) = \kappa(K,x)(u_K(x)\cdot x)^{-(n+1)}$
  - $d\mu_K(x) = (u_K(x)\cdot x)\, dx$ cone measure on $\partial K$
  - $u_K(x)$ outer unit normal vector

# Affine classification theorems

**Theorem (Haberl & Parapatits: JAMS 2014, AJM 2016)**

$\mathrm{Z} : \mathcal{P}^n_{(0)} \to \mathbb{R}$ *is a measurable and* $\mathrm{SL}(n)$ *invariant valuation*

$$\iff$$

$\exists\, c_0, c_1, c_2 \in \mathbb{R}$*:*

$$\mathrm{Z}(P) = c_0\, V_0(P) + c_1\, V_n(P) + c_2\, V_n(P^*)$$

*for every* $P \in \mathcal{P}^n_{(0)}$.

- $\mathcal{P}^n_{(0)}$ convex polytopes in $\mathbb{R}^n$ with $0 \in \operatorname{int} P$

# Affine Classification Theorems

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$\mathrm{Z} : \mathcal{K}^n_{(0)} \to \mathbb{R}$ *is an upper semicontinuous and* $\mathrm{SL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c_0, c_1, c_2 \in \mathbb{R}$ *and* $\zeta \in \mathrm{Conc}[0,\infty)$*:*

$$\mathrm{Z}(K) = c_0\, V_0(K) + c_1\, V_n(K) + c_2\, V_n(K^*) + \int_{\partial K} \zeta(\kappa_0(K,x))\, d\mu_K(x)$$

*for every* $K \in \mathcal{K}^n_{(0)}$.

**Definition**

$$\zeta \in \mathrm{Conc}[0,\infty) \;\Leftrightarrow\; \zeta : [0,\infty) \to [0,\infty) \text{ concave, } \lim_{t\to 0} \zeta(t) = \lim_{t\to\infty} \frac{\zeta(t)}{t} = 0$$

# Translation invariant Valuations

**Theorem (Homogeneous decomposition)**

$\mathrm{Z}:\mathcal{K}^n\to\mathbb{Y}$ *is a continuous, translation invariant valuation*

$$\Longrightarrow$$

$\exists\ \mathrm{Z}_0,\dots,\mathrm{Z}_n:\mathcal{K}^n\to\mathbb{Y}$ *continuous, translation invariant valuations s.t.* $\mathrm{Z}_i$ *is i-homogeneous and*

$$\mathrm{Z}=\mathrm{Z}_0+\cdots+\mathrm{Z}_n$$

- $\mathbb{Y}$ real topological vector space
- Hadwiger 1945; McMullen, Meier, Spiegel 1977
- Canonical simplex decomposition

# Translation invariant Valuations

**Theorem (Polynomiality)**

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{Y}$ *continuous, m-homogeneous, translation invariant valuation*

$$\Longrightarrow$$

$\exists$ *continuous, symmetric map* $\bar{\mathrm{Z}} : (\mathcal{K}^n)^m \to \mathbb{Y}$ *translation invariant and Minkowski additive in each variable s.t.*

$$\mathrm{Z}(t_1 K_1 + \cdots + t_k K_k) = \sum_{i_1,\dots,i_k=0}^{m} \binom{m}{i_1 \dots i_k} t_1^{i_1} \dots t_k^{i_k} \, \bar{\mathrm{Z}}(K_1 [i_1], \dots, K_k [r_k])$$

*for* $K_1, \dots, K_k \in \mathcal{K}^n$ *and* $t_1, \dots, t_k \geqslant 0$.

# Translation invariant Valuations

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*for* $K_1, \dots, K_k \in \mathcal{K}^n$ *and* $t_1, \dots, t_k \geqslant 0$.

## Corollary

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{Y}$ *continuous, 1-homogeneous, translation invariant valuation*

$$\Longrightarrow$$

$\mathrm{Z}$ *is Minkowski additive*

# Translation invariant Valuations

**Theorem (Klain & Schneider)**

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation invariant, simple valuation*

$$\Longleftrightarrow$$

$\exists\, c \in \mathbb{R}$ *and an odd continuous function* $g : \mathbb{S}^{n-1} \to \mathbb{R}$ *s.t.*

$$\mathrm{Z}(K) = c\, V_n(K) + \int_{\mathbb{S}^{n-1}} g(u)\, dS(K, u)$$

*for* $K \in \mathcal{K}^n$.

- $S(K, \cdot)$ surface area measure of $K$

# Translation invariant Valuations

**Theorem (Hadwiger 1957)**

$\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$ *is an n-homogeneous, translation invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c \in \mathbb{R}$ *s.t.* $\mathrm{Z}(P) = c\, V_n(P)$ *for* $P \in \mathcal{P}^n$.

# Translation invariant Valuations

**Theorem (Hadwiger 1957)**

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$$\iff$$

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**Theorem (McMullen 1980)**

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous,* $(n-1)$*-homogeneous, translation invariant valuation*

$$\iff$$

$\exists\, f \in C(\mathbb{S}^{n-1})$ *s.t.* $\mathrm{Z}(K) = \int_{\mathbb{S}^{n-1}} f(u)\, dS(K,u)$ *for* $K \in \mathcal{K}^n$.

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**Theorem (Alesker 2001)**

*The linear span of the valuations* $K \mapsto V(K[m], C_{m+1}, \dots, C_n)$ *with* $C_{m+1}, \dots, C_n \in \mathcal{K}^n$ *is dense in the space of continuous, m-homogeneous, translation invariant valuations on* $\mathcal{K}^n$.

# Rigid motion invariant Valuations

**Theorem (Hadwiger 1952)**

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, rigid motion invariant valuation*

$$\iff$$

$\exists \;\; c_0, c_1, \dots, c_n \in \mathbb{R}$ *:*

$$\mathrm{Z}(K) = c_0\, V_0(K) + \cdots + c_n\, V_n(K)$$

*for every* $K \in \mathcal{K}^n$*.*

- $V_0(K), \dots, V_n(K)$ intrinsic volumes of $K$
- $V_n$ $n$-dimensional volume
- $2\, V_{n-1}(K) = S(K)$ surface area
- $V_0(K)$ Euler characteristic

# Intrinsic Volumes

- $K$ convex body with smooth boundary

$$V_i(K) = \frac{\binom{n}{i}}{n v_{n-i}} \int_{\mathbb{S}^{n-1}} s_i(K,u)\, du = \frac{\binom{n}{i}}{n v_{n-i}} \int_{\mathrm{bd}\, K} H_{n-i-1}(K,x)\, dx$$

- Steiner formula

$$V_n(K + t\, B^n) = \sum_{j=0}^{n} t^{n-j} v_{n-j} V_j(K)$$

- Crofton Formula

$$V_i(K) = \int_{Graff(n,i)} V_0(K \cap E)\, d\mu_i(E) = \int_{Gr(n,i)} V_i(K|E)\, d\nu_i(E)$$

# Outline of Hadwiger's Proof

- Reduction to simple valuations:

**Theorem (Hadwiger)**

$\mathrm{Z}_s : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, rigid motion invariant, simple valuation*

$$\iff$$

$\exists\, c \in \mathbb{R}:\quad \mathrm{Z}_s(K) = c\, V_n(K)$ *for every* $K \in \mathcal{K}^n$.

  - Induction on dimension
  - Restrict $\mathrm{Z}$ to convex bodies in a hyperplane $H$:

$$\mathrm{Z}(K) = c_0\, V_0(K) + \cdots + c_{n-1}\, V_{n-1}(K)$$

  - Rigid motion invariance $\Rightarrow$ $c_0, \dots, c_{n-1}$ do not depend on $H$
  - $\mathrm{Z}_s : \mathcal{K}^n \to \mathbb{R}$, defined as $\mathrm{Z}_s = \mathrm{Z} - c_0\, V_0 - \cdots - c_{n-1}\, V_{n-1}$, is simple.
  - Theorem $\Rightarrow$ $\mathrm{Z}_s = c\, V_n \Rightarrow \mathrm{Z} = c_0\, V_0 + \cdots + c_n\, V_n$

# Outline of Hadwiger's Proof

**Theorem (Hadwiger)**

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, rigid motion invariant, simple valuation*

$$\Longleftrightarrow$$

$\exists\, c \in \mathbb{R}$ : $\mathrm{Z}(K) = c\, V_n(K)$ *for every* $K \in \mathcal{K}^n$.

- Orthogonal cylinders: $K_1 \subset E_1$ and $K_2 \subset E_1^\perp = E_2$ with $\mathbb{R}^n = E_1 \times E_2$
  - $K_1 \mapsto \mathrm{Z}(K_1 + K_2)$ continuous, rigid motion invariant valuation on $E_1$
  - $\mathrm{Z}(K_1 + K_2) = \sum_{i=0}^k c_i(K_2)\, V_i(K_1)$
  - $\mathrm{Z}(K_1 + K_2) = \sum_{i=0}^k \sum_{j=0}^{n-k} c_{ij}\, V_i(K_1)\, V_j(K_2)$
  - $\mathrm{Z}$ simple $\Rightarrow \mathrm{Z}(s\,[0,1]^k \times t\,[0,1]^{n-k}) = s^k t^{n-k}\, \mathrm{Z}([0,1]^n)$
  - $\mathrm{Z}(s\,[0,1]^k \times t\,[0,1]^{n-k}) = \sum_{i=0}^k \sum_{j=0}^{n-k} c_{ij}\, s^i\, V_i([0,1]^k)\, t^j\, V_j([0,1]^{n-k})$
  - $\mathrm{Z}(K_1 + K_2) = V_k(K_1)\, V_{n-k}(K_2) = V_n(K_1 + K_2)$
  - $\mathrm{Y} = \mathrm{Z} - V_n$ vanishes on orthogonal cylinders
- Claim: $\mathrm{Y} \equiv 0$

# Outline of Hadwiger's Proof

**Lemma (Hadwiger)**

$\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$ *is a valuation that vanishes on orthogonal cylinders* $\Rightarrow$
$\mathrm{Z}$ *is Minkowski additive*

- Canonical simplex decomposition $\Rightarrow$ $\mathrm{Z}$ is 1-homogeneous
- Polynomiality $\Rightarrow$ $\mathrm{Z}$ is Minkowski additive

# Outline of Hadwiger's Proof

**Theorem (Hadwiger)**

$\Upsilon : \mathcal{K}^n \to \mathbb{R}$ *is continuous, rigid motion invariant, Minkowski additve*

$$\Longleftrightarrow$$

$\exists \; c \in \mathbb{R}$ *:* $\quad \Upsilon(K) = c\, V_1(K)$ *for every* $K \in \mathcal{K}^n$*.*

- $\int_{\mathrm{SO}(n)} h(\vartheta K, \cdot)\, d\vartheta = \frac{V_1(K)}{V_1(B^n)}$
- $\exists$ rotation means $\frac{1}{m}\big(\vartheta_{m1} K + \cdots + \vartheta_{mm} K\big) \to \frac{V_1(K)}{V_1(B^n)} B^n$
- $\Upsilon(K) = \frac{\Upsilon(B^n)}{V_1(B^n)} V_1(K)$

**Corollary**

$\Upsilon : \mathcal{K}^n \to \mathbb{R}$ *is continuous, rigid motion invariant, Minkowski additive, simple* $\Rightarrow \Upsilon$ *vanishes identically.*

# Application: Principal Kinematic Formula

For $K, L \in \mathcal{K}^n$,

$$\int_{\phi \in \overline{\mathrm{SO}(n)}} V_0(K \cap \phi L)\, d\phi = \sum_{i=0}^{n} \frac{v_i\, v_{n-i}}{\binom{n}{i}\, v_n}\, V_i(K)\, V_{n-i}(L)$$

- $d\phi$ normalized Haar measure on $\overline{\mathrm{SO}(n)}$
- Blaschke, Chern, Hadwiger, Santaló, ...

# Application: Principal Kinematic Formula

For $K, L \in \mathcal{K}^n$,

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- $d\phi$ normalized Haar measure on $\overline{\mathrm{SO}(n)}$
- Blaschke, Chern, Hadwiger, Santaló, ...
- **Proof.** $\mathrm{Z}(K, L) = \int_{\phi \in \overline{\mathrm{SO}(n)}} V_0(K \cap \phi L)\, d\phi$
  - $\mathrm{Z}(K, \cdot), \mathrm{Z}(\cdot, L)$ continuous valuations on $\mathcal{K}^n$
  - $\mathrm{Z}(K, \cdot), \mathrm{Z}(\cdot, L)$ rigid motion invariant

# Application: Principal Kinematic Formula

For $K, L \in \mathcal{K}^n$,

$$\int_{\phi \in \overline{\mathrm{SO}(n)}} V_0(K \cap \phi L)\, d\phi = \sum_{i=0}^{n} \frac{v_i\, v_{n-i}}{\binom{n}{i}\, v_n}\, V_i(K)\, V_{n-i}(L)$$

- $d\phi$ normalized Haar measure on $\overline{\mathrm{SO}(n)}$
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$$\Rightarrow \quad \mathrm{Z}(K, L) = \sum_{i,j=0}^{n} c_{ij}\, V_i(K)\, V_j(L)$$

Determine $c_{ij}$ by choosing suitable bodies! □

# Abstract Hadwiger Theorem

**Theorem (Alesker: Annals 1999, GAFA 2007)**

*For a compact subgroup $G$ of $\mathrm{SO}(n)$, the space of continuous, $G$ and translation invariant valuations on $\mathcal{K}^n$ is finite dimensional.*

$$\Longleftrightarrow$$

*$G$ acts transitively on $\mathbb{S}^{n-1}$.*

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$$\Longleftrightarrow$$

*$G$ acts transitively on $\mathbb{S}^{n-1}$.*

- $\mathrm{U}(n)$ invariance (Alesker: GAFA 2001, Fu: JDG 2006, Bernig & Fu: Annals 2001, Wannerer: JDG 2014, AiM 2014)
- $\mathrm{SU}(n)$ invariance (Bernig: GAFA 2009)
- $\mathrm{G}_2$, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$ invariance (Bernig: Israel J. 2011, Bernig & Voide: Israel J. 2016)
- $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot\mathrm{U}(1)$, $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ invariance (Bernig & Solanes: JFA 2014, PLMS 2017)

# Hadwiger's Theorem

- Proof by Klain 1995
- Questions:
  - Classification of continuous, rigid motion invariant valuations $\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$?
  - Classification of continuous, rotation invariant valuations on $\mathbb{S}^{n-1}$?
  - Classification of rigid motion invariant and upper semicontinuous valuations on $\mathcal{K}^n$

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  - Classification of rigid motion invariant and upper semicontinuous valuations on $\mathcal{K}^n$

**Theorem (L.: Geom. Dedicata 2000)**

$\mathrm{Z} : \mathcal{K}^2 \to \mathbb{R}$ *is an upper semicontinuous, rigid motion invariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_0, c_1, c_2 \in \mathbb{R}$ *and* $\zeta \in \mathrm{Conc}[0, \infty)$*:*

$$\mathrm{Z}(K) = c_0\, V_0(K) + c_1\, V_1(K) + c_2\, V_2(K) + \int_{\partial K} \zeta(\kappa(K, x))\, dx$$

*for every* $K \in \mathcal{K}^2$.

# Classification of Vector Valuations

**Theorem (Hadwiger & Schneider 1971)**

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}^n$ *is a rotation equivariant, translation covariant, continuous valuation*

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$\exists\ c_1, \dots, c_{n+1} \in \mathbb{R}$ :

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- $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}^n$ is translation covariant $\Longleftrightarrow \exists\, \mathrm{Z}^0 : \mathcal{K}^n \to \mathbb{R}$ : $\mathrm{Z}(K + x) = \mathrm{Z}(K) + \mathrm{Z}^0(K) x \quad \forall\ x \in \mathbb{R}^n,\ K \in \mathcal{K}^n$

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# Classification of Vector Valuations

**Theorem (Dan Ma & Chunna Zeng: TAMS 2017+)**

$\mathrm{Z} : \mathcal{P}_0^2 \to \mathbb{R}^2$ *is an* $\mathrm{SL}(2)$ *equivariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_1, c_2 \in \mathbb{R}$ *and a Cauchy function* $\zeta : [0, \infty) \to \mathbb{R}$:

$$\mathrm{Z}(P) = c_1\, m_3(P) + c_2\, e(P) + h_\zeta(P)$$

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*for every* $P \in \mathcal{P}_0^n$.

- $P = [0, v_1, \dots, v_r]$
- $e(P) = v_1 + v_r$
- $h_\zeta(P) = \sum_{i=2}^{r} \frac{\zeta(\det(v_{i-1}, v_i))}{\det(v_{i-1}, v_i)}\,(v_i - v_{i-1})$

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**Theorem (Dan Ma & Chunna Zeng: TAMS 2017+)**

*For* $n \geqslant 3$*, the map* $\mathrm{Z} : \mathcal{P}_0^n \to \mathbb{R}^n$ *is an* $\mathrm{SL}(n)$ *equivariant valuation*

$$\iff$$

$\exists\ c \in \mathbb{R}$*:*

$$\mathrm{Z}(P) = c\, m_{n+1}(P)$$

*for every* $P \in \mathcal{P}_0^n$.

# Classification of Vector Valuations

**Theorem (L. 2002; Haberl & Parapatits AJM 2016)**

$\mathrm{Z} : \mathcal{P}^2_{(0)} \to \mathbb{R}^2$ *is a measurable and* $\mathrm{SL}(n)$ *equivariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_1, c_2 \in \mathbb{R}$:

$$\mathrm{Z}(P) = c_1\, m_3(P) + c_2\, \rho_{\pi/2}\, m_3(P^*)$$

*for every* $P \in \mathcal{P}^2_{(0)}$.

**Theorem (L. 2002; Haberl & Parapatits: AJM 2016)**

*For* $n \geqslant 3$, *a map* $\mathrm{Z} : \mathcal{P}^n_{(0)} \to \mathbb{R}^n$ *is an* $\mathrm{SL}(n)$ *equivariant and measurable valuation*

$$\Longleftrightarrow$$

$\exists\ c \in \mathbb{R}$:

$$\mathrm{Z}(P) = c\, m_{n+1}(P)$$

*for every* $P \in \mathcal{P}^n_{(0)}$.

# Classification of Matrix Valuations

**Theorem (L.: DMJ 2003; Haberl & Parapatits: AiM 2017)**

*For* $n \geqslant 3$, $\mathrm{Z} : \mathcal{P}^n_{(0)} \to \mathbb{T}^2$ *is an* $\mathrm{SL}(n)$ *equivariant and measurable valuation*

$$\Longleftrightarrow$$

$\exists c_1, c_2 \in \mathbb{R}$:

$$\mathrm{Z}(P) = c_1 \,\mathrm{M}^2(P) + c_2 \,\mathrm{T}^2(P^*)$$

*for every* $P \in \mathcal{P}^n_{(0)}$.

- $\mathbb{T}^2$ symmetric $n \times n$ matrices
- $\mathrm{M}^2(P) = \int_P x \otimes x \, dx$ moment matrix of $P$
- $\mathrm{T}^2(K) = \int_{\mathbb{S}^{n-1}} u \otimes u \, dS_2(K, u)$

  LYZ tensor of $K$ (Lutwak, Yang, Zhang: DMJ 2000)
- $S_2(K, \cdot) = \frac{1}{h_K} S(K, \cdot)$ $L^2$ surface area measure

# Classification of Tensor Valuations

**Theorem (Haberl & Parapatits: AiM 2017)**

*For $n \geqslant 3$, $\mathrm{Z} : \mathcal{P}^n_{(0)} \to \mathbb{T}^r$ is an $\mathrm{SL}(n)$ equivariant and measurable valuation*

$$\Longleftrightarrow$$

$\exists c_1, c_2 \in \mathbb{R}$:

$$\mathrm{Z}(P) = c_1\, \mathrm{M}^r(P) + c_2\, \mathrm{T}^r(P^*)$$

*for every $P \in \mathcal{P}^n_{(0)}$.*

- $\mathbb{T}^r$ symmetric tensors of rank $r$ in $\mathbb{R}^n$
- $\mathrm{M}^r(K) = \int_K x \otimes \cdots \otimes x \, dx = \int_K x^r \, dx$ moment tensor of rank $r$
- $\mathrm{T}^r(K) = \int_{\mathbb{S}^{n-1}} u^r \, dS_r(K, u)$
- $S_r(K, \cdot) = h_K^{1-r}\, S(K, \cdot)$ $L^r$ surface area measure

# Classification of Tensor Valuations

**Theorem (Alesker: Annals 1999, Geom. Dedicata 1999)**

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{T}^r$ *is a rotation equivariant, translation covariant, continuous valuation*

$$\Longleftrightarrow$$

$\mathrm{Z}$ *is a linear combination of* $Q^l\, \Phi_k^{m,s}$ *with* $2l + m + s = r$.

- $\Phi_k^{m,s}(K) = \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} x^m\, u^s\, d\Theta_k(K,(x,u))$ Minkowski tensors (McMullen 1997)
- $\Theta_k(K,\cdot)$ $k$-th generalized curvature measure, $Q$ metric tensor
- Steiner formula:

$$\mathrm{M}^r(K + t\,B^n) = \sum_{j=1}^{n+r} t^{n+1-j}\, v_{n+1-j} \sum_{k\in\mathbb{N}} \Phi_{j-r+k}^{r-k,k}(K)$$

Thank you!