Banach-Mazur distance to the cube

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- *Lower bounds*: Szarek, using random spaces of Gluskin type, proved that

$$\mathcal{R}_\infty^n \geqslant c\sqrt{n}\log n.$$

Szarek's theorem

- It is more convenient to work with the dual quantity
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- Let $G_1, \ldots, G_m$ be independent standard Gaussian vectors in $\mathbb{R}^n$:
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Szarek, 1990

Let $\delta > 0$ and $m = \lfloor n^{1+\delta} \rfloor$. With positive probability, $d(X_{\mathcal{G}_m}, \ell_1^n) \geqslant c(\delta)\sqrt{n} \log n$.

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- The proof involves a precise distributional inequality on the s-numbers of random Gaussian matrices, which is a quantitative finite version of Wigner's semicircle law: if $G(\omega)$ is an $n \times n$ matrix with independent $N(0, 1/n)$ Gaussian entries, then

$$\mathbb{P}\left(\omega : c_1 k/n \leqslant s_{n-k}(G(\omega)) \leqslant c_2 k/n\right) > 1 - c_3 \exp(-c_4 k^2),$$

for all $k \leqslant n/2$, where the c_i's are absolute positive constants.

Lower bound for $\mathcal{R}_\infty^n$

Tikhomirov, 2018

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- We also consider the $n \times m$ Gaussian random matrix Γ with columns $G_1, \ldots, G_m$.
- In order to show that $\mathcal{R}_1^n \geqslant \varrho$ for some $\varrho > 1$ it is enough to show that

$$\mathbb{P}\Big(\text{there exists a cross-polytope } P \text{ such that } \varrho\mathcal{G}_m \supseteq \varrho P \supseteq \mathcal{G}_m\Big) < 1.$$

Discretization

- Assume that $G_1, \dots, G_m$ are defined on the probability space Ω. For a given $\omega \in \Omega$, if a cross-polytope P is contained in $\mathcal{G}_m(\omega)$ then, by Carathéodory's theorem, its vertices are convex combinations of at most n of the vectors $\pm G_i$, and hence

$$P = \Gamma(\omega)A(B_1^n)$$

for some $m \times n$ matrix A with the property that the support of every column of A has cardinality at most n and that every column of A has ℓ_1^n-norm at most 1.

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- We consider the class $\mathcal{A}_{m,n}$ of all $m \times n$ matrices that satisfy these conditions:

$$|\mathrm{supp}\,\mathrm{col}_i(A)| \leqslant n \quad \text{and} \quad \|\mathrm{col}_i(A)\|_1 \leqslant 1.$$

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- Then, in order to show that $\mathcal{R}_1^n \geqslant \varrho$ for some $\varrho > 1$ it is enough to show that the event

$$\mathcal{E}_1 := \text{there exists } A \in \mathcal{A}_{m,n} \text{ such that } \varrho \Gamma A(B_1^n) \supseteq \mathcal{G}_m$$

  has probability $\mathbb{P}(\mathcal{E}_1) < 1$.

Discretization

- Let $\mathcal{N}$ be the set of all matrices $A = (a_{ij})$ in $\mathcal{A}_{m,n}$ with the property that $a_{ij} \in \epsilon\mathbb{Z}$ (for some small $\epsilon > 0$ to be determined).

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Reduction

Assume that $m \leqslant n^{10}$ and $\epsilon \varrho n^2 \leqslant 1$. If $\mathcal{E}_2$ is the event

$$\mathcal{E}_2 := \text{there exists } A \in \mathcal{N} \text{ such that } 2\varrho\Gamma A(B_1^n) \supseteq \mathcal{G}_m,$$

then

$$\mathbb{P}(\mathcal{E}_1) \leqslant \mathbb{P}(\mathcal{E}_2) + 2^{-n}.$$

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Proof: Set $z = (\mathbf{1}_{|y_k| \geqslant \alpha} y_k)_{k=1}^n$ and $w = (\mathbf{1}_{|y_k| < \alpha} y_k)_{k=1}^n$.

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- This will imply that every point of $\Gamma A(B_1^n)$ is a convex combination of random vectors of two types: vectors that are sparse linear combinations of G_i's and vectors whose expected Euclidean norm is small.
- The set of the first ones has small cardinality and allows a net argument, the vectors of the second type are easier to handle because they are "short".

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- We define $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{N} \to \mathcal{N}$ as follows. If $A = (a_{ij}) \in \mathcal{N}$ then $\mathcal{F}_1(A)$ is the $m \times n$ matrix with entries $\mathbf{1}_{|a_{ij}| \geqslant \alpha} a_{ij}$ and $\mathcal{F}_2(A)$ is the $m \times n$ matrix with entries $\mathbf{1}_{|a_{ij}| < \alpha} a_{ij}$.

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- Clearly, $\Gamma(\omega)A(B_1^n) \subset 2\Gamma(\omega)\mathcal{F}(A)(B_1^{2n})$.

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Proposition

Let $\mathcal{E} = \bigcap_{A \in \mathcal{N}} \mathcal{E}_A$, where $\mathcal{E}_A$ is an event which is measurable with respect to the σ-algebra generated by the vectors G_j with $j \in \theta(A) = \bigcup_{i=1}^{n} \mathrm{supp}\,\mathrm{col}_i(A)$.

If G is a standard Gaussian vector which is independent from Γ then

$$\mathbb{P}(\mathcal{E}_2 \cap \mathcal{E}) \leqslant |\mathcal{N}| \cdot \max_{A \in \mathcal{N}} \sup_{\omega \in \mathcal{E}_A} [\mathbb{P}(\omega)]^{m-n^2},$$

where

$$\mathbb{P}(\omega) := \mathbb{P}(\{\omega' : \text{there exists } I \subset [2n] \text{ with } |I| = n \text{ such that } G(\omega') \in 4\varrho\, \mathrm{conv}\{\pm \mathrm{col}_i(\Gamma(\omega)\mathcal{F}(A)), i \in I\}).$$

Proof of the Proposition

- Let $\omega \in \mathcal{E}_2 \cap \mathcal{E}$. Since $\omega \in \mathcal{E}_2$ there exists $A = A(\omega) \in \mathcal{N}$ such that

$$G_j(\omega) \in 2\varrho\,\Gamma(\omega)A(B_1^n) \subseteq 4\varrho\,\Gamma(\omega)\mathcal{F}(A)(B_1^{2n}), \qquad 1 \leqslant j \leqslant m.$$

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- Considering only $j \notin \theta(A)$ we have

$$\mathcal{E}_2 \cap \mathcal{E} \leqslant \bigcup_{A \in \mathcal{N}} \bigcap_{j \notin \theta(A)} \bigcup_{|I|=n} \Big(\mathcal{E} \cap \{ G_j(\omega) \in 4\varrho\, \mathrm{conv}\{\pm \mathrm{col}_i(\Gamma(\omega)\mathcal{F}(A)), i \in I\} \Big).$$

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- Therefore,

$$\begin{aligned}
&\mathbb{P}(\mathcal{E}_2 \cap \mathcal{E}) \\
&\leqslant |\mathcal{N}| \cdot \max_{A\in\mathcal{N}} \mathbb{P}\Big(\bigcap_{j\notin\theta(A)} \bigcup_{|I|=n} \Big(\mathcal{E}_A \cap \{G_j(\omega) \in 4\varrho\,\mathrm{conv}\{\pm\mathrm{col}_i(\Gamma(\omega)\mathcal{F}(A)), i \in I\}\Big) \\
&\leqslant |\mathcal{N}| \cdot \max_{A\in\mathcal{N}} \Big[\sup_{\omega\in\mathcal{E}_A} \mathbb{P}\Big(\bigcup_{|I|=n} \{G(\omega') \in 4\varrho\,\mathrm{conv}\{\pm\mathrm{col}_i(\Gamma(\omega)\mathcal{F}(A)), i \in I\}\Big)\Big]^{m-|\theta(A)|}.
\end{aligned}$$

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- We shall define events $\mathcal{E}_A$ measurable with respect to the σ-algebra generated by the vectors G_j with $j \in \theta(A)$, so that $\mathcal{E} = \bigcap_{A \in \mathcal{N}} \mathcal{E}_A$ satisfies

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- By the Proposition,

$$\mathbb{P}(\mathcal{E}_2 \cap \mathcal{E}) \leqslant |\mathcal{N}| \cdot \left(\frac{1}{2}\right)^{m-n^2}.$$

Strategy of the proof

- We shall define events $\mathcal{E}_A$ measurable with respect to the σ-algebra generated by the vectors G_j with $j \in \theta(A)$, so that $\mathcal{E} = \bigcap_{A \in \mathcal{N}} \mathcal{E}_A$ satisfies

$$\mathbb{P}(\mathcal{E}) \geqslant 1 - \frac{2}{n}.$$

- On the other hand, we will have

$$\sup_{\omega \in \mathcal{E}_A} \mathbb{P}(\omega) = \sup_{\omega \in \mathcal{E}_A} \mathbb{P}\Big(\bigcup_{|I|=n} \{G(\omega') \in 4\varrho\, \mathrm{conv}\{\pm \mathrm{col}_i(\Gamma(\omega)\mathcal{F}(A)), i \in I\}\}\Big) < \frac{1}{2}.$$

- By the Proposition,

$$\mathbb{P}(\mathcal{E}_2 \cap \mathcal{E}) \leqslant |\mathcal{N}| \cdot \left(\frac{1}{2}\right)^{m-n^2}.$$

- If m is chosen large (e.g. $m = n^3$) then $\mathbb{P}(\mathcal{E}_2 \cap \mathcal{E})$ will be very small, and hence

$$\mathbb{P}(\mathcal{E}_2) \leqslant \mathbb{P}(\mathcal{E}_2 \cap \mathcal{E}) + \mathbb{P}(\Omega \setminus \mathcal{E}) \leqslant \mathbb{P}(\mathcal{E}_2 \cap \mathcal{E}) + \frac{2}{n}$$

will be also small.

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$$|\{i : n - s + 1 \leqslant i \leqslant n \text{ and } \mathrm{dist}(x_i, \mathrm{span}\{x_j : j < i\}) \leqslant \sqrt{s}\}| \geqslant \frac{s}{4}.$$

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  3. For every $I \subset [2n]$ with $|I| = n$ and $|I \cap [n]| < n - q$ we have that
  $$|\{i \in I \setminus [n] : \mathrm{dist}(\mathrm{col}_i(\Gamma(\omega)\mathcal{F}(A)), \mathrm{span}\{\mathrm{col}_j(\Gamma(\omega)\mathcal{F}(A)) : j < i\}) \leqslant \tau\sqrt{\alpha|I \setminus [n]|}\}| \geqslant (1-\delta)|I \setminus [n]|.$$

An auxiliary result

The proof of the inequality $\mathbb{P}\Big(\bigcap_{A\in\mathcal{N}}\mathcal{E}_A\Big)\geqslant 1-\frac{2}{n}$ is based on the next proposition:

Distances to linear spans

Assume that $n/2\leqslant s\leqslant n$, $1\leqslant k\leqslant s/2$, $\tau\geqslant C_1$ and $\frac{1}{k}<\delta\leqslant 1$.

Let B be an $m\times s$ matrix, of rank s, with the property that each column of B has Euclidean norm at most 1.

Define $H_i=\Gamma(\mathrm{col}_i(B))$, $1\leqslant i\leqslant s$.

For any permutation σ of $[s]$, let $\mathcal{E}_\sigma$ be the event that

$$|\{i : s-k+1\leqslant i\leqslant s : \mathrm{dist}(H_{\sigma(i)},\mathrm{span}\{H_{\sigma(j)} : j<i\}\leqslant \tau\sqrt{n-s+k}|\geqslant (1-\delta)k.$$

Then, $\mathbb{P}(\mathcal{E}_\sigma)\geqslant 1-e^{-c_2\tau^2\delta(n-s+k)k}$ and

$$\mathbb{P}\Big(\bigcap_{\sigma}\mathcal{E}_\sigma\Big)\geqslant 1-s^k e^{-c_2\tau^2\delta(n-s+k)k}.$$

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- For every $p = 0, 1, \ldots, n$ there are $\binom{n}{p}^2$ ways to choose $I \subset [2n]$ with $|I| = n$ and $|I \cap [n]| = p$.
- If $\omega \in \mathcal{E}_A$ then we have

$$\begin{aligned}\mathbb{P}(\omega) &= \mathbb{P}\Big(\bigcup_{|I|=n} \{G(\omega') \in 4\varrho\,\mathrm{conv}\{\pm \mathrm{col}_i(\Gamma(\omega)\mathcal{F}(A)), i \in I\}\}\Big)\\ &\leqslant \sum_{p=0}^{n-q-1} \binom{n}{p}^2 \sup_P \gamma_n(4\varrho P) + \sum_{p=n-q}^{n} \binom{n}{p}^2 \sup_Q \gamma_n(4\varrho Q),\end{aligned}$$

where $P = \mathrm{conv}\{\pm x_1, \ldots, \pm x_n\}$ with the property that

$$|\{i : p+1 \leqslant i \leqslant n \text{ and } \mathrm{dist}(x_i, \mathrm{span}\{x_j : j < i\} \leqslant \tau\sqrt{\alpha(n-p)})\}| \geqslant (1-\delta)(n-p)$$

and $Q = \mathrm{conv}\{\pm x_1, \ldots, \pm x_n\}$ with the property that for every permutation σ of $[n]$

$$|\{i : n-s+1 \leqslant i \leqslant n \text{ and } \mathrm{dist}(x_{\sigma(i)}, \mathrm{span}\{x_{\sigma(j)} : j < i\} \leqslant \sqrt{s})\}| \geqslant \frac{s}{4}.$$

The last thing that one has to estimate is the Gaussian measure of cross-polytopes of "type P" and "type Q". The starting point is the next lemma.

Lemma 1

Let $P=\mathrm{conv}\{\pm x_1,\dots,\pm x_n\}$ be a cross-polytope and set

$$d_i=\mathrm{dist}(x_i,\mathrm{span}\{x_j : j<i\}),\qquad 2\leqslant i\leqslant n.$$

Let $1\leqslant r\leqslant n$ and consider the cross-polytope $P'=\mathrm{conv}\{\pm y_1,\dots,\pm y_n\}$, where $y_i=x_i$ if $1\leqslant i\leqslant r$ and $y_{r+1},\dots,y_n$ are mutually orthogonal vectors with $|y_i|=d_i$, which are also orthogonal to $\mathrm{span}\{x_i : i\leqslant r\}$. Then,

$$\gamma_n(P)\leqslant\gamma_n(P').$$

Lemma 2

Let $P = \mathrm{conv}\{\pm x_1, \ldots, \pm x_n\}$ be a cross-polytope such that, for some $1 \leqslant r < n$ and $h > 0$,

$$\mathrm{dist}(x_i, \mathrm{span}\{x_j : j < i\}) \leqslant h, \qquad i = r+1, \ldots, n.$$

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- **Proof.** By Lemma 1 we may assume that $\mathrm{span}\{x_1, \ldots, x_r\} = \mathrm{span}\{e_1, \ldots, e_r\}$ and $x_i = he_i$ for all $i > r$.
- If $G = (g_1, \ldots, g_n)$ is a Gaussian vector, then $G \in P$ implies that

$$\sum_{i=r+1}^{n} |g_i| \leqslant h.$$

- Therefore,

$$\gamma_n(P) = \mathbb{P}(G \in P) \leqslant \frac{1}{(2\pi)^{\frac{n-r}{2}}} \frac{(2h)^{n-r}}{(n-r)!} \leqslant \left(\frac{eh}{n-r}\right)^{n-r}.$$

Lemma 3

Let $P = \mathrm{conv}\{\pm x_1, \ldots, \pm x_n\}$ be a symmetric cross-polytope with the property that, for some $1 \leqslant p < n$, $\delta \in (0, 1/2)$ and $h > 0$,

$$|\{i : p+1 \leqslant i \leqslant n \text{ and } \mathrm{dist}(x_i, \mathrm{span}\{x_j : j < i\} \leqslant h)\}| \geqslant (1-\delta)(n-p).$$

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Lemma 4

Let $Q = \mathrm{conv}\{\pm x_1, \ldots, \pm x_n\}$ be a symmetric cross-polytope with the property that, for some $1 \ll s \leqslant n$ and for every permutation σ of $[n]$

$$|\{i : n-s+1 \leqslant i \leqslant n \text{ and } \mathrm{dist}(x_{\sigma(i)}, \mathrm{span}\{x_{\sigma(j)} : j < i\} \leqslant \sqrt{s})\}| \geqslant \frac{s}{4}.$$

Then,

$$\gamma_n(Q) \leqslant 2e^{-cs}.$$

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- Recall that we want to have $\mathbb{P}(\omega) < \frac{1}{2}$ and

$$\mathbb{P}(\omega) = \mathbb{P}\Big(\bigcup_{|I|=n} \{G(\omega') \in 4\varrho\, \mathrm{conv}\{\pm \mathrm{col}_i(\Gamma(\omega)\mathcal{F}(A)), i \in I\}\}\Big)$$

$$\leqslant \sum_{p=0}^{n-q-1} \binom{n}{p}^2 \sup_P \gamma_n(4\varrho P) + \sum_{p=n-q}^{n} \binom{n}{p}^2 \sup_Q \gamma_n(4\varrho Q),$$

and now we have upper bounds for $\gamma_n(4\varrho P)$ and $\gamma_n(4\varrho Q)$, which however depend on ϱ; this will give additional restrictions, involving ϱ, so that we will get $\mathbb{P}(\omega) < \frac{1}{2}$.

- We choose $\delta = \frac{1}{\log n}$. After the computations we have additional restrictions with ϱ: roughly,

$$s \geqslant q \log n, \qquad \varrho\sqrt{s} \leqslant n, \qquad n^2 \varrho\tau\sqrt{\alpha} \leqslant q^{5/2}.$$

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- We choose $\tau \simeq \log n \cdot \max\{\sqrt{n/q}, \sqrt{n/(q^2\alpha)}\}$ and

$$\varrho = \min\left(\frac{n}{\sqrt{s}}, \frac{1}{\log n}\frac{q^2}{n^{5/2}\sqrt{\alpha}}, \frac{1}{\log n}\frac{q^{7/2}}{n^{5/2}}\right).$$

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- This choice gives $\varrho \geqslant n^{5/9} \log^{-b} n$.