Classical operators on weighted Banach spaces of entire functions

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Joint work with
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Aim of the talk

To study the dynamics of the operators:

**Differentiation:** \( Df := f' \)

**Integration:** \( Jf(z) := \int_0^z f(\xi)d\xi, \ z \in \mathbb{C} \)

**Hardy operator:** \( Hf(z) := \frac{1}{z} \int_0^z f(\xi)d\xi, \ z \in \mathbb{C} \)

on weighted Banach spaces of entire functions.

- \( D, J \) and \( H \) are continuous on \((H(\mathbb{C}), co)\), where \( co \) denotes the compact-open topology.
- \( DJf = f \) and \( JDf(z) = f(z) - f(0) \ \forall f \in H(\mathbb{C}), \ z \in \mathbb{C} \).
Dynamics on operators

Given a Banach space $X$, 

$$\mathcal{L}(X) := \{ T : X \to X \text{ linear and continuous} \}.$$ 

Given $T \in \mathcal{L}(X)$, the pair $(X, T)$ is a linear dynamical system.

**Definitions**

- Let $x \in X$. The orbit of $x$ under $T$ is the set 
  
  $$\text{Orb}(x, T) := \{ x, Tx, T^2x, \ldots \} = \{ T^n x : n \geq 0 \}.$$ 

- $x \in X$ is a periodic point if $\exists n \in \mathbb{N}$ such that $T^n x = x$. 

Dynamics on operators

Given a Banach space $X$ and $T \in \mathcal{L}(X)$, it is said that:

**Definitions**

- $T$ **topologically mixing** $\iff \forall U, V \neq \emptyset$ open, $\exists n_0 : T^n U \cap V \neq \emptyset$ for all $n \geq n_0$.
- $T$ **hypercyclic** $\iff \exists x \in X$, $\text{Orb}(T, x) := \{x, Tx, T^2x, \ldots \}$ is dense in $X \Rightarrow X$ SEPARABLE!!!

**Definition (Godefroy, Shapiro, 1991)**

$T$ is **chaotic** if

- $T$ has a dense set of periodic points,
- $T$ is hypercyclic.
Dynamics on operators

Given a Banach space $X$ and $T \in \mathcal{L}(X)$, it is said that:

### Definitions

- $T$ power bounded $\iff \sup_n \|T^n\| < \infty$
- $T$ Cesàro power bounded $\iff \sup_n \left\| \frac{1}{n} \sum_{k=1}^{n} T^k \right\| < \infty$
- $T$ mean ergodic $\iff$

$$\forall x \in X, \ \exists P_x := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} T^k x \in X$$

- $T$ uniformly mean ergodic $\iff$

$$\left\{ \frac{1}{n} \sum_{k=1}^{n} T^k \right\}_n$$

converges in the operator norm.
Classical results

Mac Lane (1952)

$D : H(\mathbb{C}) \to H(\mathbb{C})$ is hypercyclic, i.e.,

$$\exists f_0 \in H(\mathbb{C}) : \forall f \in H(\mathbb{C}), \ \exists (n_k)_k \subseteq \mathbb{N} \text{ such that}$$

$$f_0^{(n_k)} \to f \text{ uniformly on compact sets.}$$

Proposition

The integration operator $J : H(\mathbb{C}) \to H(\mathbb{C})$ is not hypercyclic.
\( v : \mathbb{C} \to ]0, \infty[ \) is a weight if it is continuous, radial, i.e. \( v(z) = v(|z|) \), \( v(r) \) is non-increasing on \([0, \infty[\) and \( \lim_{r \to \infty} r^m v(r) = 0 \ \forall \ m \in \mathbb{N} \).

**Definition**

Given a weight \( v \), the weighted Banach spaces of entire functions:

\[
H_v^\infty := \{ f \in H(\mathbb{C}) : \| f \|_v := \sup_{z \in \mathbb{C}} v(z)|f(z)| < \infty \}
\]

\[
H_v^0 := \{ f \in H(\mathbb{C}) : \lim_{|z| \to \infty} v(z)|f(z)| = 0 \}.
\]

Given \( a \in \mathbb{R}, \ \alpha > 0 \), consider \( v_{a,\alpha}(z) := |z|^a e^{-\alpha |z|} \), for \( |z| \geq r_0 \), and the spaces \( H_{a,\alpha}^\infty \) and \( H_{a,\alpha}^0 \). For \( a = 0 \), denote them by \( H_\alpha^\infty \) and \( H_\alpha^0 \).

- \( f \in H_\alpha^\infty \iff \exists C > 0 : |f(z)| \leq Ce^{\alpha |z|} \ \forall z \in \mathbb{C} \).
- \( H_\alpha^\infty \cong \ell_\infty \) and \( H_\alpha^0 \cong c_0 \) (Lusky).
- \( \mathcal{P} \) are dense in \( H_\alpha^0 \) but the monomials are not a Schauder basis.
Lemma

Assume \( T : (H(\mathbb{C}), co) \to (H(\mathbb{C}), co) \) continuous and \( T(\mathcal{P}) \subseteq \mathcal{P} \). TFAE:

(i) \( T(H_v^\infty) \subseteq H_v^\infty \),

(ii) \( T : H_v^\infty \to H_v^\infty \) is continuous,

(iii) \( T(H_v^0) \subseteq H_v^0 \),

(iv) \( T : H_v^0 \to H_v^0 \) is continuous.

If this holds, \( \| T \|_{\mathcal{L}(H_v^\infty)} = \| T \|_{\mathcal{L}(H_v^0)} \) and \( \sigma_{H_v^\infty}(T) = \sigma_{H_v^0}(T) \), where \( \sigma_X(T) \) := \( \{ \lambda \in \mathbb{C} : T - \lambda I \) has no inverse }.

Harutyunyan, Lusky: The continuity of \( D \) and \( J \) on \( H_v^\infty(\mathbb{C}) \) is determined by the growth or decline of \( v(r)e^{\alpha r} \) for some \( \alpha > 0 \) in an interval \([r_0, \infty[\).
If \( \nu(r) = r^a e^{-\alpha r} \) (\( \alpha > 0, \ a \in \mathbb{R} \)) for \( r \geq r_0 \) : \( \|z^n\|_{a,\alpha} \approx (n+a)(n+a) \), with equality for \( a = 0 \).

**Proposition**

For \( a > 0 \):

\[
\|D^n\|_{a,\alpha} = \mathcal{O} \left( n! \left( \frac{e\alpha}{n-a} \right)^{n-a} \right) \quad \text{and} \quad n! \left( \frac{e\alpha}{n+a} \right)^{n+a} = \mathcal{O}(\|D^n\|_{a,\alpha})
\]

For \( a \leq 0 \):

\[
\|D^n\|_{a,\alpha} \approx n! \left( \frac{e\alpha}{n+a} \right)^{n+a}
\]

and the equality holds for \( a = 0 \).
Proposition

For every $\alpha > 0$ and $a \in \mathbb{R}$, the spectrum $\sigma_{a,\alpha}(D) = \alpha \overline{D}$.

Proposition

Let $\nu$ be a weight such that $D$ is continuous on $H_{\nu}^\infty(\mathbb{C})$ and that $\nu(r)e^{\alpha r}$ is non increasing for some $\alpha > 0$. If $|\lambda| < \alpha$, the operator $D - \lambda I$ is surjective on $H_{\nu}^\infty(\mathbb{C})$ and on $H_{\nu}^0(\mathbb{C})$ and it even has a continuous linear right inverse

$$K_\lambda f(z) := e^{\lambda z} \int_0^z e^{-\lambda \xi} f(\xi) d\xi, \quad z \in \mathbb{C}$$

In particular, this is satisfied by the weight $\nu_{a,\alpha}(r) = r^a e^{-\alpha r}$ for $r$ big enough (proved by Atzmon, Brive (2006), in the case $a = 0$).
Proposition

For the weight $\nu(r) = r^a e^{-\alpha r}$ ($\alpha > 0$, $a \in \mathbb{R}$) for $r$ big enough, we have:

- $\|J^n\|_{a,\alpha} \cong 1/\alpha^n$, with the equality for $a = 0$,
- $\sigma_{a,\alpha}(J) = (1/\alpha) \mathbb{D}$,
- $J - \lambda I$ is not surjective on $H^\infty_{a,\alpha}$ or $H^0_{a,\alpha}$ if $|\lambda| \leq 1/\alpha$. 
Theorem
For \( \nu \) an arbitrary weight, the Hardy operator \( H : H^\infty_\nu(\mathbb{C}) \rightarrow H^\infty_\nu(\mathbb{C}) \) is continuous with norm \( \|H\|_\nu = 1 \). Moreover, \( H^2(H^\infty_\nu(\mathbb{C})) \subset H^0_\nu(\mathbb{C}) \) and \( H^2 \) is compact. Therefore, \( \sigma(H) = \{\frac{1}{n}\}_N \cup \{0\} \). If the integration operator \( J : H^\infty_\nu(\mathbb{C}) \rightarrow H^\infty_\nu(\mathbb{C}) \) is continuous, then \( H \) is compact.

Remark
For the weight \( \nu(r) = \exp(-(\log r)^2) \):
- \( J \) is not continuous on \( H^\infty_\nu(\mathbb{C}) \) (Harutyunyan, Lusky)
- \( H : H^\infty_\nu(\mathbb{C}) \rightarrow H^0_\nu(\mathbb{C}) \), \( H : H^0_\nu(\mathbb{C}) \rightarrow H^0_\nu(\mathbb{C}) \), are compact (Lusky).
Introduction
Classical results
Weighted Banach spaces of holomorphic functions
Continuity, norms and spectrum
Dynamics of $D$ and $J$ on $H^\infty_{a,\alpha}$ and $H^0_{a,\alpha}$

Hypercyclicity

Theorem (Bonet, 2009)

$D : H^0_{a,\alpha} \to H^0_{a,\alpha}$ satisfy:

- $0 < \alpha < 1 \implies D$ is not hypercyclic and has no periodic point different from 0.
- $\alpha = 1 \implies$ if $a < 1/2$, then $D$ is topologically mixing, and if $a \geq 1/2$, $D$ is not hypercyclic. It has no periodic point different from 0 iif $a \geq 0$.
- $\alpha > 1 \implies D$ is chaotic and topologically mixing.
Mean ergodicity

**Remark**

\[ T \in \mathcal{L}(X) \text{ Cesàro bounded and } P(d) = 0 \text{ for every } d \in D, \ D \subseteq X \text{ dense} \implies T \text{ mean ergodic}. \]

**Proposition**

Let \( T = D \) or \( T = J \) and assume \( T : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C}) \) is continuous. TFAE:

(i) \( T : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C}) \) is uniformly mean ergodic,

(ii) \( T : H^0_v(\mathbb{C}) \to H^0_v(\mathbb{C}) \) is uniformly mean ergodic,

(iii) \( \lim_{N \to \infty} \frac{||T + \cdots + T^N||_v}{N} = 0. \)

Moreover, if \( 1 \in \sigma_v(T) \), then \( T \) is not uniformly mean ergodic.
Mean ergodicity

Theorem (Lin)
Let $T \in \mathcal{L}(X)$ such that $\|T^n/n\| \to 0$. Then,

$$T \text{ uniformly mean ergodic } \iff (I - T)X \text{ is closed}.$$  

Theorem (Lotz)
Let $T \in \mathcal{L}(H^\infty_\alpha)$ such that $\|T^n/n\| \to 0$. Then,

$$T \text{ mean ergodic } \iff T \text{ uniformly mean ergodic}.$$
Mean ergodicity

Theorem

Let $v(r) = e^{-\alpha r}$, $r \geq 0$. $D$ is power bounded if and only if $\alpha < 1$. It is uniformly mean ergodic on $H_\alpha^\infty(\mathbb{C})$ and $H_\alpha^0(\mathbb{C})$ if $\alpha < 1$, not mean ergodic if $\alpha > 1$, and it is not mean ergodic on $H_1^\infty(\mathbb{C})$ and not uniformly mean ergodic on $H_1^0(\mathbb{C})$.

Let $v(r) = e^{-\alpha r}$, $r \geq 0$. $J$ is never hypercyclic and it is power bounded if and only if $\alpha \geq 1$. If $\alpha > 1$, $J$ is uniformly mean ergodic on $H_\alpha^\infty(\mathbb{C})$ and $H_\alpha^0(\mathbb{C})$ and it is not mean ergodic on these spaces if $\alpha < 1$. If $\alpha = 1$, then $J$ is not mean ergodic on $H_1^\infty(\mathbb{C})$, and mean ergodic but not uniformly mean ergodic on $H_1^0(\mathbb{C})$.

For every weight $v$, $H$ is power bounded, not hypercyclic and uniformly mean ergodic on $H_v^\infty(\mathbb{C})$. 
### Summary

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Dynamics on weighted Bergman spaces

Given a weight $\nu$, $1 \leq p \leq \infty$, and $1 \leq q < \infty$,

$$B_{p,q}(\nu) := \left\{ f \in \mathcal{H}(\mathbb{C}) : \|f\|_{p,q,\nu} := \left(2\pi \int_0^\infty r \nu(r)^q M_p(f, r)^q \, dr\right)^{1/q} < \infty \right\}$$

$$B_{p,\infty}(\nu) := \left\{ f \in \mathcal{H}(\mathbb{C}) : \|f\|_{p,\nu} := \sup_{r>0} \nu(r) M_p(f, r) < \infty \right\}$$

$$B_{p,0}(\nu) := \left\{ f \in \mathcal{H}(\mathbb{C}) : \lim_{r \to \infty} \nu(r) M_p(f, r) = 0 \right\}.$$
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