Algebra and Geometry in Functional Analysis

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(Universidad de Almería)

VIII Encuentro de la Red de Análisis Funcional

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Our goal

We describe briefly the research activities of the group FQM-194 (Junta de Andalucía) and, more precisely, those developed under the following project:

Estructuras no asociativas y Análisis Funcional. Finitud en términos de operadores en módulos y espacios de Banach

Referencia: MTM2010-17687
Investigador Principal: Amín Kaidi
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Contents

1. Extremal structure and unitary Banach algebras
2. Absolute-valued algebras and division normed algebras
3. Linear preserver problems
4. Applications of Measure Theory to copulas and singular functions. Dependence Models
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Introduction
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Linear preserver problems
Applications of Measure Theory to copulas and singular functions
Nonlinear Functional Analysis and Partial Differential Equations

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Minkowski-Carathéodory 1911

Let $K$ be a compact and convex subset of $\mathbb{R}^n$. Then, every element of $K$ can be expressed as a convex combination of $m \leq n + 1$ extreme points of $K$.

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Let $K$ be a compact and convex subset of a Hausdorff locally convex space. Then $K$ is the closed convex hull of its extreme points.
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Preliminaries in $C^*$-algebras

Russo-Dye 1966

The closed unit ball of every (complex and unital) $C^*$-algebra coincides with the closed convex hull of its unitary elements.


Every point $a$ of a $C^*$-algebra, with $\|a\| < 1$, is the average of a finite number of unitary elements. In particular, the open unit ball of each $C^*$-algebra is contained in the convex hull of the set of its unitary elements.

As we have already said, the above results are valid for complex unital $C^*$-algebras. Indeed, there are real $C^*$-algebras with only two unitary elements (it is the case of $C([0, 1], \mathbb{R})$).
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The concept of unitary element can be extended to any unital Banach algebra $\mathcal{A}$. Specifically, an element $a \in \mathcal{A}$ is said to be unitary if it is invertible and $\|a\| = \|a^{-1}\| = 1$.

The algebra $\mathcal{A}$ is called unitary if the convex hull of its unitary elements is dense in the unit ball of $\mathcal{A}$.

The Russo-Dye theorem shows that any (complex and unital) $C^*$-algebra is unitary.

Unitary Banach algebras have been studied (among others) by Cowie (under the name of convex-transitive Banach algebras), Hansen, Pedersen and more recently by Becerra, Burgos, Kaidi and Rodríguez.
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Some Questions

- Analysis of the validity of the Russo-Dye Theorem for real unital $C^*$-algebras

Let $X$ be a Banach space, $G_X$, $N_X$ and $E_{L(X)}$ the group of surjective isometries, the semigroup of nice operators (linear and continuous mappings $T : X \to X$ such that $T^*e^*$ is an extreme point for every extreme point $e^*$ of the unit ball of $X^*$) and the set of extreme points of the unit ball of $L(X)$, respectively. It is clear that $G_X \subseteq N_X \subseteq E_{L(X)}$.

- Study of the possibility of equality in any of the preceding inclusions.
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- Study of the possibility of equality in any of the preceding inclusions.
Finally, let $M$ be a metric space, $X$ a Banach space and $U(M, X)$ the space of bounded and uniformly continuous functions from $M$ into $X$ equipped with its uniform norm.

- Analysis of the interaction between the extremal structure of the unit ball of $U(M, X)$ and the properties of the pair $(M, X)$.
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- Analysis of the interaction between the extremal structure of the unit ball of $U(M, X)$ and the properties of the pair $(M, X)$
Absolute-valued algebras

These algebras represent one of the successful research lines in non-associative algebras theory.

Absolute-valued algebras are defined as those real or complex algebras $A$ satisfying $\|xy\| = \|x\| \|y\|$, for all $x, y \in A$ and a given norm $\|\cdot\|$ on $A$.

Working lines:
- To advance on classification of absolute valued algebras.
- Characterization of absolute valued algebras verifying non-trivials identities.
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A still unsolved old question is that of nonassociative extension of the Gelfand-Mazur theorem, namely if any division normed (nonassociative) algebra must be finite-dimensional (which would imply dimension 1 in the complex case, and 1, 2, 4, 8 in the real one, by a theorem of Bott and Milnor). This problem was explicitly posed by Wright in 1953, who in the same paper gave a partial affirmative answer proving that division absolute-valued algebras are finite-dimensional. The case of (two-sided) division noncomplete normed complex algebras as well as that of division (even complete) normed real algebras remaining open. In the converse direction, Cuenca and Rodríguez given examples of infinite-dimensional one-sided division absolute-valued algebras over the field of real numbers.
Precedents

The Banach–Stone type theorems began with a result of Banach included in his famous book from 1932:

If $X$ and $Y$ are two compact metric spaces, then every surjective linear isometry $T : C(X, \mathbb{R}) \rightarrow C(Y, \mathbb{R})$ can be represented by a weighted composition operator of the form $T(f) = \tau \cdot (f \circ \varphi)$ for all $f \in C(X, \mathbb{R})$, where $\varphi : Y \rightarrow X$ is a homeomorphism and $\tau : Y \rightarrow \{-1, 1\}$ is a continuous function.

In 1937, Stone improved Banach’s theorem by establishing it for arbitrary compact Hausdorff topological spaces. This new version, valid for spaces $C(X)$ of real or complex continuous functions, is today known as the Banach–Stone theorem.
We must point out that the Banach–Stone theorem has inspired a great number of generalizations and extensions. The paper *Variations on the Banach-Stone theorem* by Garrido and Jaramillo provides us with a first approach to this subject. The different algebraic-topological structures of $C(X)$ motivate the study of the morphisms that identify them. This structures allow us to raise the problem of obtaining a Banach–Stone type representation for algebra homomorphisms, ring homomorphisms and vector lattice homomorphisms, among others.

From a more general perspective, in the last years there exists an enormous interest to characterize those applications leaving invariant a certain set, function or relation. These problems are known as *linear preserver problems*. 

Juan Carlos Navarro Pascual (Universidad de Almería)
Maps between spaces of Lipschitz functions

For this kind of (usually linear) maps it is studied the Banach–Stone type representation. The aim is to get a description in terms of weighted composition operators for (non-necessarily surjective, scalar and vector valued) linear isometries, codimension 1 isometries, local isometries, 2-local isometries, vector lattice homomorphisms, biseparating linear maps and surjective maps that preserve multiplicatively the spectrum, the peripheral range, etc.
Jordan homomorphisms

Another research line in linear preserver problems focuses on the description of transformations preserving some kind of invertibility.

Kaplansky posed one of the most famous problems:

Let $\phi : A \to B$ be a unital surjective linear map between two semisimple Banach algebras which preserves invertibility, that is:

$$\phi(a) \in B^{-1} \quad \text{for all} \quad a \in A^{-1}.$$ 

Is it true that $\phi$ is a Jordan homomorphism?

A linear map $\phi : A \to B$ is called a Jordan homomorphism if:

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Preserving inverses

Several authors gave partial answers to this problem, but it still remains open even in the setting of general C*-algebras.

In 1949 Hua proved that every unital additive map between two fields which preserves inverses is an homomorphism or anti-homomorphism (hence a Jordan homomorphism).

In general we say that a linear map \( \phi : A \rightarrow B \) between two Banach algebras preserve inverses or that strongly preserve the invertibility if \( \phi(a^{-1}) = \phi(a)^{-1} \) for all \( a \in A^{-1} \).

The definition can be adapted to other types of invertibility. In fact, Hua’s result has been extended to maps between Banach algebras which strongly preserve generalized invertibility.
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The definition can be adapted to other types of invertibility. In fact, Hua’s result has been extended to maps between Banach algebras which strongly preserve generalized invertibility.
Let $A$ be a Banach algebra and $a \in A$. It is said that $b \in A$ is a:

- **generalized inverse** of $a$ if $aba = a$ and $bab = b$
- **Drazin inverse** of $a$ if $ab = ba$, $bab = b$ and $a^k ba = a^k$ for some $k$ in $\mathbb{N} \cup \{0\}$
- **group inverse** of $a$ if $ab = ba$, $bab = b$ and $aba = a$.

If $A$ has also an involution $*$, it is said that $b \in A$ is a

- **Moore-Penrose inverse** of $a \in A$ if $bab = b$, $aba = a$, $(ab)^* = ab$ and $(ba)^* = ba$.

The Drazin inverse and the group inverse are unique. The Moore-Penrose inverse is unique in the C*-algebra setting.
Generalizations of the invertibility

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We are concerned with linear or additive maps between Banach algebras or C*-algebras that strongly preserve one of the above types of generalized invertibility. Results have been achieved about:

- Characterization of additive maps between Banach algebras that strongly preserve Drazin (resp. group) invertibility.
- Characterization of linear maps between C*-algebras that strongly preserve generalized invertibility in the triple sense.
- Characterization of linear maps between C*-algebras that strongly preserve Moore-Penrose invertibility, where the domain is unital and has real rank zero or essential socle.
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In addition to the working lines we have described so far, all under the project MTM2010-17687, we will refer to two other activities of some other members of the group FQM-194.
In recent years, there has been an increasing interest in the construction of multivariate stochastic models that take into account the dependence among two or more variables, with numerous applications in finance, actuarial science, environment studies, health science, etc. Since any doubly stochastic measure is a copula, they are a very useful tool for studying the relationship between random variables and, therefore, they have several applications in modelling and simulation process. Further, functions with a "peculiar" behaviour, as the singular functions (i.e., monotone increasing and continuous functions whose derivatives are null almost everywhere), can satisfy the "self-similarity" property and they often appear with graphs as a fractal set.
With typical techniques of these theories, together with new ones from the Theory of Measure and Probability, and using properties relating to the ergodicity or the fractality and representation systems of the real numbers in [0,1], among others, we investigate theoretical, as well as, practical problems structured in four blocks:

- Construction of copulas
- Measure-preserving functions and Copulas with fractal support
- Representation systems and singular functions
- Singular functions in fractals and Interpolation
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Motivated by different models arising from Physics, we study mathematical models in the field of nonlinear partial differential equations, including some of the new trends of the last years. Namely we consider quasilinear elliptic equations and systems, with natural growth at the gradient, critical nonlinearities or singularities.
Mainly we study qualitative properties (existence, uniqueness, regularity, bounds, ... ) by using a complete variety of tools from Nonlinear Functional Analysis:

- Critical Point Theory: Minimization, Mountain Pass, Morse Index, etc.,
- Topological Methods: A priori Estimates, Fixed Point Theorems, Leray-Schauder Degree and Bifurcation,
- Monotone Operators: Sub and Supersolutions.
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Finally I would like to introduce my colleagues in the group and appreciate their contribution in this document:

- Enrique de Amo
- José Carmona
- Juan Fernández
- Antonio Jiménez
- Amín Kaidi
- Antonio Carlos Márquez
- Antonio Morales
- Miguel Ángel Navarro
- Maribel Ramírez
- Mari Gracia Sánchez-Lirola
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