Dinámica lineal de $C_0$-semigrupos

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Let $X$ be a Banach space, and $L(X)$ be the set of linear and continuous operators from $X$ to $X$.

**Definition**

A family of linear and continuous operators $\{T\}_{t \geq 0}$ in $L(X)$ is said to be a semigroup on $X$ if the following are verified:

1. $T_0 = I$.
2. $T_t T_s = T_{t+s}$ for all $t, s \geq 0$.

A semigroup in $L(X)$ is strongly continuous (or a $C_0$-semigroup) if in addition

3. $\lim_{t \to s} T_t x = T_s x$ for all $x \in X, s \geq 0$.

A semigroup in $L(X)$ is uniformly continuous if in addition

3'. $\lim_{t \to s} T_t x = T_s x$ for all $s \geq 0$ uniformly on $X$.

**Example**

The translation semigroup on $L^p(\mathbb{R}^+), 1 \leq p < \infty$, defined as $\{T_t\}_{t \geq 0}$ with $T_t f(s) := f(s + t)$ is a $C_0$-semigroup.


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**Theorem**

\( \{ T \}_{t \geq 0} \) is a uniformly continuous semigroup if and only if there is \( A \in L(X) \) such that \( T_t = e^{tA}, t \geq 0 \).

**Definition**

The **infinitesimal generator** \( A : Dom(A) \subseteq X \rightarrow X \) of a semigroup \( \{ T \}_{t \geq 0} \) on \( X \), is the operator

\[
Ax := \lim_{h \rightarrow 0^+} \frac{T_hx - x}{h}
\]

defined for every \( x \) where this limit exists.

The generator of a \( C_0 \)-semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.

**Example**

The translation semigroup on \( X = L^p(\mathbb{R}^+) \), \( 1 \leq p < \infty \), has \( D \) as infinitesimal generator with \( Dom(D) = \{ u \in X : u \text{ absolutely continuous, and } u' \in X \} \).
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Hypercyclicity

Definition

\( \{ T_t \}_{t \geq 0} \) is **hypercyclic** if there exists some \( x \in X \) such that

\[
\text{Orb}(\{ T_t \}_{t \geq 0}, x) := \{ T_t x : t \geq 0 \} \text{ is dense in } X.
\]

Theorem

If there is a hypercyclic \( C_0 \)-semigroup on \( X \), then

- \( X \) must be infinite-dimensional, and
- \( X \) must be separable.

This notion coincides with transitivity (Birkhoff '20):

Definition

\( \{ T_t \}_{t \geq 0} \) is **transitive** if for any pair of nonempty open sets \( U, V \subset X \) there exists \( t_0 > 0 \) such that \( T_{t_0} U \cap V \neq \emptyset \).
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$x \in X$ is a **periodic point** for $\{T_t\}_{t \geq 0}$ if there is some $t_0 > 0$ such that $T_{t_0}x = x$.

**Definition**

$\{T_t\}_{t \geq 0}$ is a **chaotic $C_0$-semigroup** in the sense of Devaney if

1. it is hypercyclic,
2. it has a dense set of periodic points, and
3. it has sensitive dependence on the initial conditions.

The third condition can be deduced from the first condition.
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Definition

\( \{ T_t \}_{t \geq 0} \) has **sensitive dependence on the initial conditions** if

\[
\exists \delta > 0 \forall x \in X, \varepsilon > 0
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\[
\exists y \in X, t_0 > 0 \text{ such that }
\]

\[
\| x - y \| < \varepsilon \& \| T_{t_0} x - T_{t_0} y \| > \delta
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Hypercyclicity \( \Rightarrow \) Sensitive dependence on the initial conditions

If \( \{ T_t \}_{t \geq 0} \) is hypercyclic, then \( \lim_{t \to \infty} ||T_t|| = \infty \).
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**Hypercyclicity** $\Rightarrow$ **Sensitive dependence on the initial conditions**

If $\{T_t\}_{t \geq 0}$ is hypercyclic, then $\lim_{t \to \infty} \|T_t\| = \infty$.  

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Li-Yorke chaos

Can we find many of these points?

**Definition**

A $C_0$-semigroup $\{T_t\}_{t \geq 0}$ is said to be **Li-Yorke chaotic** if there exists an uncountable subset $\Gamma \subset X$, called the **scrambled** set, such that for every pair $x, y \in \Gamma$ of distinct points we have that

$$\liminf_{t \to \infty} \| T_t x - T_t y \| = 0 \text{ and } \limsup_{t \to \infty} \| T_t x - T_t y \| > 0.$$ 

**Every hypercyclic $C_0$-semigroup is Li-Yorke chaotic**

Fix a hypercyclic vector $x \in X$ for $\{T_t\}_{t \geq 0}$ and consider $\Gamma := \{ \lambda x ; |\lambda| \leq 1 \}$ as a scrambled set.
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**Every hypercyclic $C_0$-semigroup is Li-Yorke chaotic**

Fix a hypercyclic vector $x \in X$ for $\{T_t\}_{t \geq 0}$ and consider $\Gamma := \{\lambda x; |\lambda| \leq 1\}$ as a scrambled set.
Definition (Upper density)

Given a subset $B \subset \mathbb{R}^+_0$ we define its upper density as

$$\text{Dens}(B) := \limsup_{t \to \infty} \frac{\mu(B \cap [0, t])}{t},$$

where $\mu$ stands for the Lebesgue measure on $\mathbb{R}^+_0$. 
Definition (Distributional Chaos)

A $C_0$-semigroup $\{T_t\}_{t \geq 0}$ in $L(X)$ with a scrambled set $S$ is **distributionally chaotic** on $S$ if there is some $\delta > 0$ such that for each $\epsilon > 0$ and each pair $x, y \in S$ of distinct points we have

$$\overline{\text{Dens}}(\{s \geq 0 : \|T_s x - T_s y\| > \delta\}) = 1 \quad \text{and} \quad (1)$$

$$\overline{\text{Dens}}(\{s \geq 0 : \|T_s x - T_s y\| < \epsilon\}) = 1. \quad (2)$$

If the scrambled set $S$ is dense on $X$, then $\{T_t\}_{t \geq 0}$ is said to be **densely distributionally chaotic**.
Distributional chaos

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If the scrambled set $S$ is dense on $X$, then $\{T_t\}_{t \geq 0}$ is said to be densely distributionally chaotic.
Distributional chaos consists on the existence of certain $\delta > 0$ such that for every pair $x, y \in S$ and for all $\varepsilon > 0$,

there are $m, n \in \mathbb{R}^+$ large enough such that for every $x, y \in S$, $x \neq y$ such that

1. the measure of the set of indexes in $[0, m]$ such that $||T_t x - T_t y|| \geq \delta$ is greater than $(1 - \varepsilon)m$

2. the measure of the set of indexes in $[0, n]$ such that $||T_t x - T_t y|| < \varepsilon$ is greater than $(1 - \varepsilon)n$
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1. the measure of the set of indexes in $[0, m]$ such that $\| T_t x - T_t y \| \geq \delta$ is greater than $(1 - \varepsilon)m$
2. the measure of the set of indexes in $[0, n]$ such that $\| T_t x - T_t y \| < \varepsilon$ is greater than $(1 - \varepsilon)n$
Definition (Distributional irregular vector)

Taking \( y = 0 \), if there is some \( x \in X \) such that these conditions hold for any \( \delta, \epsilon > 0 \),

\[
\text{Dens}\left(\{s \geq 0 : \|T_s x - T_s y\| > \delta\}\right) = 1 \quad \text{and} \quad \text{Dens}\left(\{s \geq 0 : \|T_s x - T_s y\| < \epsilon\}\right) = 1.
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The operator case

Hypercyclicity and Devaney chaos have been intensively studied during the last 20 years.

Distributional chaos, Li-Yorke chaos and distributionally irregular vectors have been introduced for the operator case by

- Martínez, Oprocha & Peris in JMAA (2009).
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The dynamics of the translation $C_0$-semigroup

Consider the space $L^p_\rho(\mathbb{R}^+)$, $1 \leq p < \infty$.

**Rolewicz’69**
If $\rho(s) := a^{-s}$, $a > 1$, then the translation $C_0$-semigroup is hypercyclic.

**Desch, Schappacher & Webb ’97**
\[ \liminf_{s \to \infty} \rho(s) = 0 \iff \text{Hypercyclicity.} \]

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\[ \int_0^\infty \rho(s) < \infty \iff \text{Devaney chaos.} \]

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If there is $B \subset \mathbb{R}$ has positive upper density and $\int_B \rho(s) < \infty \Rightarrow \text{Distributional chaos.}$

Distributionally chaos does not imply Devaney chaos nor hypercyclicity.
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J.A. Conejero 2012
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If there is a hypercyclic vector $x \in X$ for $\{T_t\}_{t \geq 0}$, then $x$ is a hypercyclic vector for any single operator $T_{t_0}$ with $t_0 > 0$.

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Based on a result by Conejero, Martínez & Peris ’12 on the existence of certain chaotic operators with a prescribed set of periods we have:

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Definition (Upper density)

Given a subset $B \subset \mathbb{N}$ we define its upper density as

$$\overline{\text{Dens}}(B) := \limsup_{n \to \infty} \frac{|B \cap [1, n]|}{n},$$

where $|\cdot|$ stands for the counting measure on $\mathbb{N}$. 
How to determine Devaney chaos?

Kitai '82, Gethner & Shapiro'91, Bès, Peris '99

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Theorem (Eigenvalue criterion for Devaney chaos)

Suppose that the sets

\[ X_0 := \text{span}\{x \in X : \lambda \text{ with } \Re(\lambda) < 0, T_t x = e^{\lambda t} x, \forall t \geq 0\}, \]

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**Theorem**

If the following conditions hold:

1. There exists a dense subset $X_0 \subset X$ with $\lim_{t \to \infty} T_t x = 0$, for each $x \in X_0$, and
2. there exists some $\lambda \in \sigma_p(A)$ with $\Re(\lambda) > 0$,

then $\{T_t\}_{t \geq 0}$ is distributionally chaotic. Furthermore, it admits a dense distributionally irregular manifold.

**Remark**

If a $C_0$-semigroup verifies the eigenvalue criterion, then it is Devaney chaotic and distributionally chaotic. Furthermore, it admits a dense distributionally irregular manifold.
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**Theorem**

If the following conditions hold:

1. There exists a dense subset $X_0 \subset X$ with $\lim_{t \to \infty} T_t x = 0$, for each $x \in X_0$, and
2. there exists some $\lambda \in \sigma_p(A)$ with $\Re(\lambda) > 0$,

then $\{T_t\}_{t \geq 0}$ is distributionally chaotic. Furthermore, it admits a dense distributionally irregular manifold.

**Remark**

If a $C_0$-semigroup verifies the eigenvalue criterion, then it is Devaney chaotic and distributionally chaotic. Furthermore, it admits a dense distributionally irregular manifold.
How to determine distributional chaos?

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If a $C_0$-semigroup verifies the eigenvalue criterion, then it is Devaney chaotic and distributionally chaotic. Furthermore, it admits a dense distributionally irregular manifold.
A pair of examples:

**Desch, Schappacher & Webb '97**

Consider the following PDE in $L^2(\mathbb{R}^+, \mathbb{C})$:

\[ u_t(x, t) = au_{x,x}(x, t) + bu_x(x, t) + cu(x, t), \]
\[ u(0, t) = 0 \text{ for } t \geq 0, \]
\[ u(x, 0) = f(x) \text{ for } x \geq 0, \text{ with some } f \in X \]

$Au := au_{x,x}(x, t) + bu_x(x, t) + cu(x, t)$,

and $a, b, c > 0$ with $c < b^2/2a < 1$.

The solution semigroup (which has $A$ as infinitesimal generator) is Devaney chaotic, and it is also distributionally chaotic.
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The solution semigroup (which has $A$ as infinitesimal generator) is Devaney chaotic, and it is also distributionally chaotic.
Conejero & Mangino '10

Consider the following PDE in $L^2(\mathbb{R}^+, \mathbb{C})$:

\[
\begin{align*}
  u_t(x, t) &= u_{xx}(x, t) + bu_x(x, t) + cu(x, t), \\
  u(0, t) &= 0 \quad \text{for } t \geq 0, \\
  u(x, 0) &= f(x) \quad \text{for } x \geq 0, \text{ with some } f \in X \\
\end{align*}
\]

$Au := u_{xx}(x, t) + bu_x(x, t) + cu(x, t),$

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The solution semigroup (which has $A$ as infinitesimal generator) is Devaney chaotic, and it is also distributionally chaotic.
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The solution semigroup (which has $A$ as infinitesimal generator) is Devaney chaotic, and it is also distributionally chaotic.
Devaney chaos is sometimes proved in a constructive way or reducing the problem to the translation $C_0$-semigroup.
Example 1

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \zeta(x) \frac{\partial u}{\partial x} + h(x)u \\
u(0,x) &= f(x)
\end{aligned}
\]  

(5)

with \( h, \zeta \) bounded and continuous functions on \( \mathbb{R}_0^+ \). If \( \zeta(x) = 1 \),

\[
T_t f(x) = \exp \left( \int_x^{x+t} h(s) \, ds \right) f(x+t), \text{ for } f \in X.
\]

If we define \( \rho(x) = \exp \left( -\int_0^x h(s) \, ds \right) \) and \( \phi(f)(x) = (\rho(x))^{1/p} f(x) \), we have:

\[
\begin{array}{ccc}
L_p^\rho(\mathbb{R}_0^+, \mathbb{C}) & \xrightarrow{T_t} & L_p^\rho(\mathbb{R}_0^+, \mathbb{C}) \\
\phi & \downarrow & \phi \\
L_p(\mathbb{R}_0^+, \mathbb{C}) & \xrightarrow{T_t} & L_p(\mathbb{R}_0^+, \mathbb{C})
\end{array}
\]  

(6)

If \( h(x) \) is constant and equal to 1 Devaney and dist. chaos.
If \( h(x) = -1 \) if \( x \in [n^2, n^2 + 1[ \) for some \( n \in \mathbb{N} \), and \( h(x) = 1 \) elsewhere, dist. chaos but not Devaney chaos.
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\[
L^p_\rho(\mathbb{R}^+_0, \mathbb{C}) \xrightarrow{\tau_t} L^p_\rho(\mathbb{R}^+_0, \mathbb{C}) \quad \phi \\
\downarrow \quad \downarrow \phi
\]

\[
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Example 2

Take \( \rho : [0,1] \rightarrow \mathbb{R}^+ \) continuous s.t. there are \( M \geq 1 \) and \( w \in \mathbb{R} \) verifying

\[
\rho(x) \leq Me^{\omega t} \rho(e^{\gamma t} x) \quad \text{for all } x \in [0,1] \text{ and } t > 0.
\]

\[S_t f(x) = f(e^{\gamma t} x) \quad \text{for } t \geq 0 \text{ and } f \in L_p^\rho([0,1], \mathbb{C}), 1 \leq p < \infty.\]

- \( X_0 \) as the set of continuous functions on \([0,1]\) with \( g(0) = 0 \).
- \( \lim_{t \to \infty} S_t x = 0 \) for every \( x \in X_0 \).
- \( \|S_t\|^{-1}_p \rho \) can be integrated respect to \( t \) on \([0,\infty)\).
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Example 3

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \xi(x) \frac{\partial u}{\partial x} + h(x)u \\
u(0,x) &= f(x)
\end{aligned}
\]

(7)

Take the case \(\xi(x) = \gamma x, \gamma < 0\) and \(h\) continuous. If there is \(\delta > 0\) s.t. \(\Re(h(x)) \geq 0\) for \(0 \leq x \leq \delta\), then

\[T_t f(x) = \exp \left( \int_0^t h(e^{\gamma(t-r)}x)f(e^{\gamma t}x) \right) \text{ for } t \geq 0,\]

Define \(\rho(x) = \exp\{1/\gamma\int_x^1 (h(s)/s)ds\}\) and set \(\phi(f)(x) = (\rho(x))^{1/p}f(x)\):

\[
\begin{align*}
L_p^\rho([0,1], \mathbb{C}) &\xrightarrow{S_t} L_p^\rho([0,1], \mathbb{C}) \\
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L^p([0,1], \mathbb{C}) &\xrightarrow{T_t} L^p([0,1], \mathbb{C})
\end{align*}
\]

(8)

then we have distributional chaos.
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\[
T_{t}f(x) = \exp \left( \int_{0}^{t} h(e^\gamma(t-r)) x(f(e^\gamma r x) \right) \text{ for } t \geq 0,
\]

Define \( \rho(x) = \exp \left\{ \frac{1}{\gamma} \int_{x}^{1} (h(s)/s) ds \right\} \) and set \( \phi(f)(x) = (\rho(x))^{1/p}f(x) \):

\[
L_p^p([0, 1], \mathbb{C}) \xrightarrow{S_t} L_p^p([0, 1], \mathbb{C}) \xrightarrow{\phi} L_p^p([0, 1], \mathbb{C}) \\
L^p([0, 1], \mathbb{C}) \xrightarrow{T_t} L^p([0, 1], \mathbb{C})
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\end{array}
\]

\[
L^p([0,1], \mathbb{C}) \xrightarrow{T_t} L^p([0,1], \mathbb{C})
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then we have distributional chaos.
Example 4

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \zeta(x) \frac{\partial u}{\partial x} + h(x)u \\
u(0, x) &= f(x)
\end{aligned}
\] (9)

Take the case \( \zeta(x) = 1, \ h(x) = \frac{kx}{1 + x^k}. \)

\[ T_t f(x) = \frac{1 + (x + t)^k}{1 + x^k} f(x + t) \text{ for } x, \geq 0, \ t \geq 0, \]

If we define \( \rho(x) = \frac{1}{1 + x^k} \) and set \( \phi(f)(x) = (\rho(x))^{1/p} f(x), \) we have:

\[
\begin{array}{ccc}
L_p^p(\mathbb{R}_0^+, \mathbb{C}) & \xrightarrow{\tau_t} & L_p^p(\mathbb{R}_0^+, \mathbb{C}) \\
\phi & & \phi \\
L^p(\mathbb{R}_0^+, \mathbb{C}) & \xrightarrow{T_t} & L^p(\mathbb{R}_0^+, \mathbb{C})
\end{array}
\] (10)

Again, as \( \int_0^\infty \rho(x)dx < \infty, \) then it is distributionally chaotic.
Example 4

\[ \begin{cases} \frac{\partial u}{\partial t} = \zeta(x) \frac{\partial u}{\partial x} + h(x)u \\ u(0,x) = f(x) \end{cases} \]

Take the case \( \zeta(x) = 1, \) \( h(x) = \frac{kx}{1+x^k}. \)

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If we define \( \rho(x) = \frac{1}{1+x^k} \) and set \( \phi(f)(x) = (\rho(x))^{1/p} f(x), \) we have:

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Take the case \( \zeta(x) = 1, \ h(x) = \frac{ks}{1 + x^k} \).

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T_t f(x) = \frac{1 + (x + t)^k}{1 + x^k} f(x + t) \text{ for } x, t \geq 0.
\]

If we define \( \rho(x) = \frac{1}{1 + x^k} \) and set \( \phi(f)(x) = (\rho(x))^{1/p} f(x) \), we have:

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L_p^\rho(\mathbb{R}_0^+, \mathbb{C}) \xrightarrow{T_t} L_p^\rho(\mathbb{R}_0^+, \mathbb{C})
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\[T_tf(x) = \frac{1 + (x + t)^k}{1 + x^k} f(x + t) \] for \( x, \geq 0, t \geq 0 \),

If we define \( \rho(x) = \frac{1}{1 + x^k} \) and set \( \phi(f)(x) = (\rho(x))^{1/p} f(x) \), we have:

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L^p_\rho(\mathbb{R}^+_0, \mathbb{C}) & \xrightarrow{\tau_t} L^p_\rho(\mathbb{R}^+_0, \mathbb{C}) \\
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Again, as \( \int_0^\infty \rho(x)dx < \infty \), then it is distributionally chaotic.
Some open questions:

Last example was already known to be Devaney chaotic (El Mourchid ’06) but it cannot be applied the Eigenvalue criterion.

There is a weaker version by Desch, Schappacher & Webb ’97, and restated by El Mourchid.

**Theorem (El Mourchid ’06)**

Let $X$ be a complex separable Banach space, and let $\{T_t\}_{t \geq 0}$ be a $C_0$-semigroup on $X$ with infinitessimal generator $(A, D(A))$. Assume that there are $a < b$ and continuous functions $f_j : [a, b] \rightarrow X, j \in J$, such that

1. $f_j(s) \in \ker(isI - A)$ for every $s \in [a, b], j \in J$,
2. $\text{span}\{ f_j(s); s \in [a, b], j \in J \}$ is dense in $X$.

Then the semigroup $\{T_t\}_{t \geq 0}$ is Devaney chaotic.
Some open questions:

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Let $X$ be a complex separable Banach space, and let $\{T_t\}_{t \geq 0}$ be a $C_0$-semigroup on $X$ with infinitesimal generator $(A, D(A))$. Assume that there are $a < b$ and continuous functions $f_j : [a, b] \to X, j \in J$, such that

1. $f_j(s) \in \ker(isl - A)$ for every $s \in [a, b], j \in J$,
2. $\text{span}\{f_j(s); s \in [a, b], j \in J\}$ is dense in $X$.

Then the semigroup $\{T_t\}_{t \geq 0}$ is Devaney chaotic.
Some open questions:

Last example was already known to be Devaney chaotic (El Mourchid ’06) but it cannot be applied the Eigenvalue criterion.

There is a weaker version by Desch, Schappacher & Webb ’97, and restated by El Mourchid.

**Theorem (El Mourchid ’06)**

Let $X$ be a complex separable Banach space, and let $\{T_t\}_{t \geq 0}$ be a $C_0$-semigroup on $X$ with infinitessimal generator $(A,D(A))$. Assume that there are $a < b$ and continuous functions $f_j : [a,b] \to X, j \in J$, such that

1. $f_j(s) \in \ker(isI - A)$ for every $s \in [a,b], j \in J$,
2. $\text{span}\{f_j(s); s \in [a,b], j \in J\}$ is dense in $X$.

Then the semigroup $\{T_t\}_{t \geq 0}$ is Devaney chaotic.
Some open questions:

**Question 1**
Do these hypothesis imply the existence of distributional chaos?

This question could have a positive answer, but it is also unknown whether Devaney chaos imply distributional chaos on $C_0$-semigroups.

**Question 2**
Are there examples of Devaney chaotic $C_0$-semigroups which are not distributionally chaotic?
Some open questions:

**Question 1**
Do these hypothesis imply the existence of distributional chaos?

This question could have a positive answer, but it is also unknown whether Devaney chaos imply distributional chaos on $C_0$-semigroups.

**Question 2**
Are there examples of Devaney chaotic $C_0$-semigroups which are not distributionally chaotic?
Thank you very much for your attention.