Topological groups and $C(X)$ spaces with ordered bases

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Outline

1. \(\Sigma\)-bases in topological groups
2. Boundedly complete sets and long \(\Sigma\)-bases
3. Existence of proper long \(\Sigma\)-bases on \(C_c([0, \omega_1])\)
Outline

1. Σ-bases in topological groups
   - θ-bases and quasi-θ-bases
   - Σ-bases and $C_c(X)$ with Σ-base
**Definition**

A topological group $G$ is said to have a $G$-base if there is a base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighborhoods of the identity $e$ in $G$ such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$.

- Metrizable topological group $\implies$ $G$-base.
- Fréchet-Urysohn topological group with a $G$-base $\implies$ metrizable (Grabriyelyan ..., Fundamenta Math. 2015).

**Definition**

A compact resolution on a topological space $X$ is a compact covering $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of $X$ such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$. If for each compact subset $K$ of $X$ there exists $K_\alpha$ such that $K \subset K_\alpha$, then $\mathcal{K}$ is a compact resolution swallowing compact subsets.
Theorem

A space $C_c(X)$ has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of (absolutely convex) neighborhoods of the origin if and only if $X$ has a compact resolution $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallowing compact subsets.
Proof.

We may suppose that there exists a compact $K$

$$U_\alpha \subset W(K, [-1, 1]) := \{ f \in C_c(X) : f(K) \subset [-1, 1] \}, \quad \text{hence}$$

$$K \subset K_\alpha := \cap_{f \in U_\alpha} f^{-1}([-1, 1]) \quad \text{and} \quad U_\alpha \subset W(K_\alpha, [-1, 1]), \quad \alpha \in \mathbb{N}^\mathbb{N}$$

There exists a compact $K_{U_\alpha}$ and $\varepsilon_\alpha > 0$ such that

$$W(K_{U_\alpha}, (-\varepsilon, \varepsilon)) \subset U_\alpha.$$  Then

$$W(K_{U_\alpha}, (-\varepsilon, \varepsilon)) \subset W(K_\alpha, [-1, 1]) \implies K_\alpha \subset K_{U_\alpha}.$$  

$$\{ K_\alpha : \alpha \in \mathbb{N}^\mathbb{N} \}$$ is a compact resolution swallowing compact sets

The converse follows from

$$W(K_{\alpha=(a_1, \ldots)}, [-a_1^{-1}, a_1^{-1}]) \subset W(K, [-\varepsilon, \varepsilon]) \quad \text{if} \quad K \subset K_\alpha, \quad a_1^{-1} < \varepsilon.$$
Corollary

If $X$ is a Polish space the $C_c(X)$ has a $\mathcal{G}$-base. Whence $C_c(\mathbb{R}^\mathbb{N})$ is a non-metrizable locally convex space with a $\mathcal{G}$-base.

Proof.

Let $\{x_n : n \in \mathbb{N}^\mathbb{N}\}$ be a dense subset of $X$, $d$ a complete metric compatible and $B(x_{am}, n^{-1})$ the closed ball of center $x_{am}$ and radius $n^{-1}$. If $\alpha := (a_n)_n \in \mathbb{N}^\mathbb{N}$ and

$$K_\alpha := \cap_{n \in \mathbb{N}^\mathbb{N}} \left[ \bigcup_{1 \leq m \leq n} B(x_{am}, n^{-1}) \right]$$

we get that $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution of $X$ swallowing compact sets.

Finally, $\mathbb{R}^\mathbb{N}$ is Polish but not hemicompact.
**Definition (Tsaban and Zdomskyy, 2009)**

A topological group $G$ has the strong Pytkeev property if there exists a sequence $\mathcal{D}$ of subsets of $G$ satisfying the property: for each neighborhood $U$ of the unit $e$ and each $A \subseteq G$ with $e \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite.

**Proposition (Gabriyelyan, Kąkol and Leiderman, 2014)**

Any topological group $G$ with the strong Pytkeev property admits a quasi-$\mathcal{G}$-base $\{U_\alpha : \alpha \in \Sigma\}$ of the identity, i.e., an ordered base of neighborhoods $\{U_\alpha : \alpha \in \Sigma\}$ of $e$ over some $\Sigma \subseteq \mathbb{N}^\mathbb{N}$.
Proposition (Banakh, 2015)

For every separable metrizable space $X$ the space $C_c(X)$ has the strong Pytkeev property; therefore such $C_c(X)$ admits a quasi-$\mathfrak{F}$-base.

Remark

Let $X$ be a separable metric space which is not a Polish space. Then $C_c(X)$ has a quasi-$\mathfrak{F}$-base but $C_c(X)$ does not admit a $\mathfrak{F}$-base.
The following is a more practical concept than quasi-$\mathcal{G}$-base.

**Definition**

If $\Sigma \subseteq \mathbb{N}^\mathbb{N}$ is an unbounded (i.e., $\sup\{\alpha(k) : \alpha \in \Sigma\} = \infty$ for some $k \in \mathbb{N}$) and directed subset of $\mathbb{N}^\mathbb{N}$, a base $\{U_\alpha : \alpha \in \Sigma\}$ of neighborhoods of the neutral element of a topological group $G$ is a $\Sigma$-base if $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$ with $\alpha, \beta \in \Sigma$.

**Theorem** (For a completely regular space $X$ are equivalent:)

1. The locally convex space $C_c(X)$ has a $\Sigma$-base of absolutely convex neighborhoods of the origin.

2. There is a compact covering $\{K_\alpha : \alpha \in \Sigma\}$ of $X$ that swallows the compact sets of $X$, with $\Sigma$ unbounded, directed and such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$ in $\Sigma$. 

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Proof.

If $U_\alpha$ is a neighborhood of the origin in $C_c(X)$ and $K$ is a compact subset of $X$ such that

$$U_\alpha \subset W(K, [-1, 1]) := \{f \in C_c(X) : f(K) \subset [-1, 1]\},$$

then

$$K \subset K_\alpha := \cap_{f \in U_\alpha} f^{-1}([-1, 1]) \quad \text{and} \quad U_\alpha \subset W(K_\alpha, [-1, 1]).$$

Let $K_{U_\alpha}$ be a compact set such that $W(K_{U_\alpha}, (-\varepsilon, \varepsilon)) \subset U_\alpha$. Then

$$W(K_{U_\alpha}, (-\varepsilon, \varepsilon)) \subset W(K_\alpha, [-1, 1]) \quad \Rightarrow \quad K_\alpha \subset K_{U_\alpha}.$$
continued proof.

If $C_c(X)$ has a $\Sigma$-base there exists a compact subset $K$ of $X$ and a $\Sigma$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that $U_\alpha \subset W(K, [-1, 1])$, for each $\alpha \in \mathbb{N}^\mathbb{N}$. Whence $\{K_\alpha : \alpha \in \Sigma\}$ is a compact covering of $X$, with $\Sigma$ unbounded and directed, such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$ in $\Sigma$, that swallows the compact sets.

To proof the converse we must to take into account that given a compact subset $K$ of $X$ and a positive real number $\varepsilon > 0$ there exists $\alpha \in \Sigma$ such that $K \subset K_\alpha$ and $a_n^{-1} < \varepsilon$, whence

$$W(K_\alpha=(a_1,\ldots), [-a_n^{-1}, a_n^{-1}]) \subset W(K, [-\varepsilon, \varepsilon]).$$
Theorem

If \((X, d)\) is a separable and not Polish, then \(C_c(X)\) admits \(\Sigma\)-base and it does not admit any \(\mathcal{G}\)-base.

Proof (only the non trivial part).

Let \(D := \{x_m : m \in \mathbb{N}\}\) dense subset in \(X\), \(\{y_n : n \in \mathbb{N}\}\) dense in \(K\) (compact), \(x_{n(p)} \in D\) with

\[
\lim_{p} x_{np} = y_n \quad \text{and} \quad d(x_{np}, y_n) < n^{-1}, \text{for each } p \in \mathbb{N},
\]

then \(K \subset \overline{\{x_{np} : (n, p) \in \mathbb{N}^2\}}\) (compact). The \(\Sigma\)-base follows from the set \(\alpha := (a_n)n \in \bigcup_{m \in \mathbb{N}\setminus\{1\}}\{1, m\}^\mathbb{N}\) with compact

\[K_\alpha := \{x_n : n \in \mathbb{N}, a_n \neq 1\}\]
2 Boundedly complete sets and long $\Sigma$-bases
- Boundedly complete subsets of $\mathbb{N}^\mathbb{N}$
- Long $\Sigma$-bases
In this section we are going to consider a special class of \( \Sigma \)-bases, which we denominate long \( \Sigma \)-bases, and study some properties of them quite close to those of \( \mathcal{G} \)-bases.

**Definition**

A subset \( \Sigma \) of \( \mathbb{N}^\mathbb{N} \) will be called boundedly complete if each bounded set \( \Delta \) of \( \Sigma \) has a bound at \( \Sigma \).

- \( \Sigma \) boundedly complete \( \implies \) \( \Sigma \) is directed.
- If \( \{ U_\alpha : \alpha \in \Sigma \} \) is an infinite base of neighborhoods of a (Hausdorff) locally convex space and \( \Sigma \) is a boundedly complete subset of \( \mathbb{N}^\mathbb{N} \) then \( \Sigma \) must be unbounded.
  (Otherwise \( \sup \{ \alpha (k) : \alpha \in \Sigma \} < \infty \) for every \( k \in \mathbb{N} \) \( \implies \) there exists \( \gamma \in \Sigma \) with \( \alpha \leq \gamma \) for every \( \alpha \in \Sigma \). Hence \( U_\gamma \subseteq \bigcap_{\alpha \in \Sigma} U_\alpha \), a contradiction.)
Example

Every cofinal subset $\Sigma$ of $\mathbb{N}^\mathbb{N}$ with respect to the partial order ‘$\leq$’ is boundedly complete.

Proof.

If $\beta(k) := \sup \{\alpha(k) : \alpha \in \Delta\} < \infty$ for every $k \in \mathbb{N}$, then $\beta := (\beta(k))_k \in \mathbb{N}^\mathbb{N}$, hence there is $\gamma \in \Sigma$ such that $\beta \leq \gamma$.

Proposition

If $X$ is a topological space with a compact covering $\{A_\alpha : \alpha \in \Sigma\}$ that swallows the compact sets indexed by a boundedly complete subset $\Sigma$ of $\mathbb{N}^\mathbb{N}$ and such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ in $\Sigma$, then $X$ is strongly dominated by a second countable space.
Compact coverings and strong domination

Proof.

Let \( T : \Sigma \rightarrow \mathcal{K}(X) \) defined by \( T(\alpha) = A_\alpha \) and let \( K \) be a compact set in \( \Sigma \).

\[
\sup \{ \alpha(k) : \alpha \in K \} < \infty, \forall k \in \mathbb{N} \implies \exists \gamma \in \Sigma, \alpha \leq \gamma, \forall \alpha \in K.
\]

\[
T(K) = \bigcup_{\alpha \in K} T(\alpha) \subseteq A_\gamma \implies B_K := \overline{T(K)} \text{ is compact}
\]

\( B := \{ B_K : K \in \mathcal{K}(\Sigma) \} \) is an increasing compact covering of \( X \) that swallows the compact sets, because

if \( P \) is compact \( \exists \delta \in \Sigma \) with \( P \subseteq T(\delta) = B_{\{\delta\}} \).

Hence \( X \) is strongly \( \Sigma \)-dominated (\( \Sigma \) separable metric). \( \square \)
Long \( \Sigma \)-bases

**Definition**

A \( \Sigma \)-base of neighborhoods of the unit element of a topological group \( G \) indexed by a boundedly complete subspace \( \Sigma \) of \( \mathbb{N}^\mathbb{N} \) will be referred to as a long \( \Sigma \)-base.

Of course, every \( \mathcal{G} \)-base of neighborhoods of the origin of a locally convex space \( E \) is a long \( \Sigma \)-base, with \( \Sigma = \mathbb{N}^\mathbb{N} \). The proof of the next theorem uses the following

**Proposition (Cascales, Orihuela, Tkachuk, 2011)**

A compact topological space \( K \) is metrizable if and only if the space \( (K \times K) \setminus \Delta \) is strongly dominated by a second countable space, where here \( \Delta := \{(x, x) : x \in K\} \).
Long $\Sigma$-bases and metrizability

**Theorem**

If a topological group $G$ has a long $\Sigma$-base $\{U_\alpha : \alpha \in \Sigma\}$ then every compact subset $K$ in $G$ is metrizable. Consequently, $G$ is strictly angelic.

**Proof.**

It is enough to show that $W := (K \times K) \setminus \Delta$ has a compact covering $\mathcal{W} := \{W_\alpha : \alpha \in \Sigma\}$ that swallows the compact sets indexed by a boundedly complete subset $\Sigma \subseteq \mathbb{N}^\mathbb{N}$ and such that $W_\alpha \subseteq W_\beta$ whenever $\alpha \leq \beta$ in $\Sigma$. We may assume that all sets $U_\alpha$ are symmetric and open. Then
continued proof.

\[ W_\alpha := \{ (x, y) \in W : xy^{-1} \notin U_\alpha \} \]

- is closed in \( K \times K \), hence \( W_\alpha \) compact. If \( Q \subseteq W \) is a compact set. Then

\[ e \notin T(Q) := \{ xy^{-1} : (x, y) \in Q \} \] (compact),

implies there exists \( U_\alpha \) such that

\[ U_\alpha \cap T(Q) = \emptyset \implies Q \subseteq W_\alpha. \]

\[ \mathcal{W} := \{ W_\alpha : \alpha \in \Sigma \} \] verifies the conditions.
**Corollary**

If there exists a family \( \{ A_\alpha : \alpha \in \Sigma \} \) made up of compact sets, indexed by a boundedly complete set \( \Sigma \) such that \( A_\alpha \subseteq A_\beta \) whenever \( \alpha \leq \beta \) and satisfying that \( \bigcup \{ A_\alpha : \alpha \in \Sigma \} = X \), then \( C_c(X) \) is strictly angelic.

**Proof.**

\( X \) is web-compact, so \( C_p(X) \) is angelic (Orihuela 1987), whence \( C_c(X) \) is angelic (by angelic lemma). To prove "strict" let \( Y = \bigcup \{ A_\alpha : \alpha \in \Sigma \} \) and \( \tau_p \) and \( \tau_c \) pointwise and the compact-open topology on \( C(Y) \).
continued proof.

(Σ boundedly complete \(\implies\) unbounded in \(\mathbb{N}^\mathbb{N}\) \(\implies\)) there exists \(k \in \mathbb{N}\) such that \(\sup \{\alpha(k) : \alpha \in \Sigma\} = \infty\). Then

\[\{U_\alpha : \alpha \in \Sigma\}, \text{ with } U_\alpha := \{f \in C(Y) : \sup_{y \in A_\alpha} |f(y)| \leq \alpha(k)^{-1}\}\]

is a long \(\Sigma\)-base of a lc topology \(\tau\) on \(C(Y)\) such that \(\tau_p \leq \tau \leq \tau_c\). By preceding Theorem every \(\tau\)-compact set in \(C(Y)\) is metrizable. Whence each compact subset \(K\) of \(C_c(X)\) is metrizable since the restriction map \(S : C_c(X) \to (C(Y), \tau)\) is continuous and \(S\) restricts itself to an homeomorphism on each compact subset \(K\) of \(C_c(X)\).
Theorem

If $C_c(X)$ has a long $\Sigma$-base of neighborhoods of the origin, then $X$ is a $C$–Suslin space. Consequently $C_c(X)$ is angelic.

Proof.

$X$ has a compact covering $\{K_\alpha : \alpha \in \Sigma\}$ swallowing compacts such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$.

Let $T : \Sigma \to K(X)$ defined by $T(\alpha) = A_\alpha$.

If $\alpha_n \in \Sigma$ and $\lim_n \alpha_n = \alpha \in \mathbb{N}^\mathbb{N}$, then there is $\gamma \in \Sigma$ with $\alpha_n \leq \gamma$ for every $n \in \mathbb{N}$.

Consequently, $\{T(\alpha_n) : n \in \mathbb{N}\} \subset A_\gamma$. Hence $x_n \in T(\alpha_n)$, $\forall n \in \mathbb{N}$, $\implies \{x_n\}_{n=1}^\infty$ has a cluster point $x$ in $X$ (contained in $A_\gamma$).

Therefore $X$ is web-compact.
A limit property in Fréchet-Urysohn topological groups

Let \( \{ U_\alpha : \alpha \in \Sigma \} \) be a long \( \Sigma \)-base in a topological group \( G \).
For every \( \alpha = (a_i)_{i \in \mathbb{N}} \in \Sigma \) and each \( k \in \mathbb{N} \), set

\[
\alpha(k) := (a_1, a_2, \ldots, a_k)
\]

\[
D_k(\alpha) := \cap \{ U_\beta : \beta \in \Sigma, \beta(k) = \alpha(k) \}.
\]

Clearly, \( \{ D_k(\alpha) \}_{k \in \mathbb{N}} \) is an increasing and \( e \in D_k(\alpha) \).

Proposition (Chasco, Martín-Peinador and Tarieladze, 2007)

Let \( \{ x_{n,k} : (n, k) \in \mathbb{N} \times \mathbb{N} \} \) a subset of a Fréchet-Urysohn topological group \( G \) such that \( \lim_n x_{n,k} = x \in G, k = 1, 2, \ldots \).
There exists two increasing sequences of natural numbers \( (n_i)_{i \in \mathbb{N}} \) and \( (k_i)_{i \in \mathbb{N}} \), such that \( \lim_i x_{n_i,k_i} = x \).
Theorem

Each Fréchet-Urysohn topological group $G$ with a long $\Sigma$-base $\{U_\alpha : \alpha \in \Sigma\}$ is metrizable.

Proof.

Assume $\exists \alpha \in \Sigma$ such that $D_k(\alpha)$ is not a neighborhood of the unit $e$ for every $k \in \mathbb{N}$.

$$e \in G \setminus D_k(\alpha) \implies \exists \{x_{n,k}\}_{n \in \mathbb{N}} \text{ in } G \setminus D_k(\alpha) \text{ converging to } e.$$ Hence exists $(n_i)_{i \in \mathbb{N}} \uparrow$ and $(k_i)_{i \in \mathbb{N}} \uparrow$ such that $\lim_i x_{n_i,k_i} = e$.

$x_{n_i,k_i} \notin D_{k_i}(\alpha) \implies \exists \beta_{k_i} \in \Sigma, \beta_{k_i}(k_i) = \alpha(k_i), x_{n_i,k_i} \notin U_{\beta_{k_i}}$.

$x_{n_i,k_i} \notin U_{\gamma}$ for every $i \in \mathbb{N}$, if $\beta_{k_i} \leq \gamma$, $i \in \mathbb{N}$. Contradiction.

For every $\alpha \in \Sigma$ choose the minimal $k_\alpha \in \mathbb{N}$ such that $D_{k_\alpha}(\alpha)$ is a neighborhood of $e$.

$\{\text{int}(D_{k_\alpha}(\alpha))\}_{\alpha \in \Sigma}$ is base of neigh. of $e$, so $G$ is metrizable.
Corollary

Let \( \{G_t\}_{t \in T} \) be a family of metrizable topological groups. Then the product \( G := \prod_{t \in T} G_t \) has a long \( \Sigma \)-base if and only if \( T \) is countable, i.e., when \( G \) is metrizable.

Proof.

Let \( e_t \) be the unit vector in \( G_t \) for \( t \in T \).

The \( \Sigma \)-product \( G_0 := \{ x = (x_t) \in G : |t \in T : x_t \neq e_t| \leq \aleph_0 \} \) is a dense Fréchet-Urysohn subgroup of \( G \) (Noble, 1970).

If \( G \) has a long \( \Sigma \)-base, then \( G_0 \) enjoys also this property. Whence \( G_0 \) is metrizable, so \( G \) is metrizable, too. The converse is clear.
Long $\Sigma$-bases in $C_p(X)$

Corollary

The space $C_p(X)$ has a long $\Sigma$-base if and only if $X$ is countable.

Proof.

Apply preceding Corollary to $\mathbb{R}^X = \overline{C_p(X)}$. 

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Topological groups and $C(X)$ spaces with ordered bases
3 Existence of proper long $\Sigma$-bases on $C_c([0,\omega_1])$
The dominating cardinal

In \((\mathbb{N}^\mathbb{N}, \leq^*)\)

- \(\alpha \leq^* \beta\) stands for the *eventual dominance preorder* defined so that \(\alpha(n) \leq \beta(n)\) for almost all \(n \in \mathbb{N}\), i.e., for all but finitely many values of \(n\).

- \(\alpha <^* \beta\) means that there exists \(m \in \mathbb{N}\) such that \(\alpha(n) < \beta(n)\) for every \(n \geq m\).

\(\omega_1\) is the first ordinal of uncountable cardinal, whose cardinality we denote by \(\aleph_1\).

ZFC model means Zermelo–Fraenkel model + axiom of choice.

**Definition**

The *dominating cardinal* \(\mathfrak{d}\) is the least cardinality for cofinal subsets of the preordered space \((\mathbb{N}^\mathbb{N}, \leq^*)\).

One has \(\aleph_1 \leq \mathfrak{d} \leq c\).
The main lemma

Lemma

If \( \aleph_1 = \emptyset \) there exists a cofinal \( \omega_1 \)-sequence \( \Gamma := \{ \beta_\kappa : \kappa < \omega_1 \} \) in \( (\mathbb{N}^\mathbb{N}, \leq^*) \) such that

1. \( \kappa_1 < \kappa_2 \) implies that \( \beta_{\kappa_1} <^* \beta_{\kappa_2} \),
2. for each \( \alpha \in \mathbb{N}^\mathbb{N} \) the subset
   \[
   \Delta_\alpha := \{ \kappa < \omega_1 : \beta_\kappa \leq^* \alpha \}
   \]
   of \([0, \omega_1)\) is countable,
3. if \( \alpha \leq^* \gamma \) then \( \Delta_\alpha \subseteq \Delta_\gamma \), and
4. every countable subset of \([0, \omega_1)\) is contained in some \( \Delta_\gamma \); in particular, \( \bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \Delta_\alpha = [0, \omega_1) \).
Example

In any ZFC model for which $\aleph_1 = d < \mathfrak{c}$ there exists a completely regular space $X$ and a compact covering $\{A_\alpha : \alpha \in \Sigma\}$ of $X$, with $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ and indexed by an unbounded, directed and boundedly complete proper subset $\Sigma$ of $\mathbb{N}^\mathbb{N}$ that swallows the compact sets of $X$.

Corollary

In any ZFC model for which $\aleph_1 = d < \mathfrak{c}$ there exists a long $\Sigma$-base of absolutely convex neighborhoods of the origin of the space $C_c([0, \omega_1])$ which is not a $\mathfrak{g}$-base.
Problem

Let $X$ be a separable metric space admitting a compact ordered covering of $X$ indexed by an unbounded and boundedly complete proper subset of $\mathbb{N}^\mathbb{N}$ that swallows the compact sets of $X$. Is then $X$ a Polish space?
References I


References II


References III


