Extended Eigenvalues for bilateral weighted shifts

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Introduction

Extended eigenvalues for Cesàro operators
Extended Eigenvalues for bilateral weighted shifts
Some open questions

Definition

If \( AX = \lambda XA \) for some \( X \neq 0 \), \( \lambda \) is called an extended eigenvalue and \( X \) an (extended) eigenoperator of \( A \).

Scott Brown (1979) and Kim, Moore and Pearcy (1979), independently.

If an operator \( A \) on a Banach space has a non-zero compact eigenoperator, then \( A \) has a nontrivial, hyperinvariant subspace.

Lomonosov (1973)

If \( A \) commutes with a non-zero compact operator then \( A \) has a non-trivial hyperinvariant subspace.
Extended eigenvalue has taken on a life of its own.

- Further results of Lomonosov-type [4, 5].
- Studies of the extended eigenvalues and eigenoperators for interesting classes of naturally occurring operators [1, 2, 3, 1, 2]

We continue this latter thread


Introduction
Extended eigenvalues for Cesàro operators
Extended Eigenvalues for bilateral weighted shifts
Some open questions

Extended eigenvalues for $C_\infty$.
Extended eigenvalues for $C_1$ and $C_0$.

Cesàro operators on $\ell^2$, $L^2[0,1]$ and $L^2[0,\infty)$

\[
(C_0 f)(n) = \frac{1}{n+1} \sum_{k=0}^{n} f(k) \quad (C_1 f)(x) = \frac{1}{x} \int_{0}^{x} f(s) \, ds
\]

\[
(C_\infty f)(x) = \frac{1}{x} \int_{0}^{x} f(s) \, ds
\]

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Question
Extended eigenvalues for Cesàro operators?

Main Theorem
The set of extended eigenvalues for $C_\infty$ it reduces to the singleton \{1\}, for $C_1$ it is the interval $(0, 1]$ and for $C_0$ is the interval $[1, \infty)$. 
Extended eigenvalues for $C_\infty$.

**Definition**

A bounded linear operator $U$ on a complex Hilbert space $H$ is a bilateral shift of multiplicity one provided that there is an orthonormal basis $(e_n)$ of $H$ such that $Ue_n = e_{n+1}$ for all $n \in \mathbb{Z}$.

**Brown-Halmos- Shields. $C_\infty$ on $L^2[0, \infty)$**

They proved that $C_\infty$ is a bounded linear operator, and they also proved that $I - C_\infty^*$ is unitarily equivalent to a bilateral shift of multiplicity one.
Theorem
Let $U$ be a bilateral shift of multiplicity one, and let $\lambda$ be a complex number with $\lambda \neq 1$. Then the equation $(I - U^*)X = \lambda X(I - U^*)$ has only the trivial solution $X = 0$.

Lemma
Let $X$ be an operator satisfying $(I - U^*)X = \lambda X(I - U^*)$, and let $\cdots, X_{-1}, X_0, X_1, X_2, \cdots$ be the rows of the matrix of $X$. Then $X_{n+1} = (\lambda U + 1 - \lambda)X_n$, for all $n \in \mathbb{Z}$. Consequently, for any $m, n \in \mathbb{N}$, $X_{m+n} = (\lambda U + 1 - \lambda)^n X_m$. In particular, if $m = 0$, $X_n = (\lambda U + 1 - \lambda)^n X_0$, for all $n \in \mathbb{N}$. 

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Extended eigenvalues
The set of extended eigenvalues for the infinite continuous Cesàro operator $C_\infty$ defined on the complex Banach space $L^p[0, \infty)$ reduces to the singleton $\{1\}$.

There exists a Schauder basis $\{e_n\}, n \in \mathbb{Z}$ on $L^q[0, \infty)$ such that $(1 - 2/qC_\infty^*)e_n = e_{n+1}$ for all $n \in \mathbb{Z}$. 
Introduction
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Tools

1. To prove that $\lambda$ is an extended eigenvalue for $T$, there is no choice but to show the existence of an operator $X_\lambda$ such that $TX_\lambda = \lambda X_\lambda T$: a) Constructively b) Using Baire Category’s theorem.

2. (Rosenblum) If $\sigma(A) \cap \sigma(B) = \emptyset$ then $X = 0$ is the only solution of the equation $AX - XB = 0$. If $\lambda$ is an extended eigenvalue then $\sigma(T) \cap \sigma(\lambda T) \neq \emptyset$.

3. Semigroup techniques were used by Biswas to discard extended values for the Volterra operator.
Extended eigenvalues for $C_1$ and $C_0$.

**Operators with rich point spectrum**

We say that an operator $T$ on a complex Banach space has rich point spectrum provided that $\text{int} \sigma_p(T) \neq \emptyset$, and that for every open disc $D \subset \sigma_p(T)$, the family of eigenvectors $\bigcup_{z \in D} \ker(T - z)$ is a total set.

**Theorem**

Let us suppose that an operator $T$ on a complex Banach space has rich point spectrum. If $\lambda$ is an extended eigenvalue for $T$ then we have $\lambda \cdot \text{int} \sigma_p(T) \subset \text{clos} \sigma_p(T)$. 
$C_0^*$ and $C_1$ have rich point spectrum.

**Proposition**

On $\ell^p$ spaces the extended eigenvalues for $C_0$ is contained on $[1, \infty)$.

**Proposition**

If $\lambda$ is an extended eigenvalue for $C_1$ on $L^p[0, 1]$ then $\lambda$ is real and $0 < \lambda \leq 1$. 
Extended eigenvalues for $C_1$ on $L^p[0, 1]$

**Theorem**

If $0 < \lambda \leq 1$ then $\lambda$ is an extended eigenvalue for the Cesàro operator $C_1$ on $L^p[0, 1]$ and a corresponding extended eigenoperator is the weighted composition operator $X_0 \in B(L^p[0, 1])$ defined by

$$(X_0 f)(x) = x^{(1-\lambda)/\lambda} f(x^{1/\lambda}).$$
Extended eigenvalues for $C_0$ on $\ell^2$

Kriete-Trutt (1974/75)

There exists a positive finite measure defined on the Borel subsets of the complex plane and supported on $\overline{D}$ and a unitary operator $U : \ell^2 \to H^2(\mu)$ such that $C_0 = U^*(I - M_z)U$, ($H^2(\mu)$ denotes the closure of the polynomials on $L^2(\mu)$).

Theorem

If $\lambda \geq 1$ then $\lambda$ is an extended eigenvalue for $I - M_z$ and a corresponding extended eigenoperator is the composition operator $X$ defined by the expression $(Xf)(z) = f\left(\frac{\lambda - 1}{\lambda} + \frac{z}{\lambda}\right)$.

Corollary

On $\ell^2$ the set of extended eigenvalues for $C_0$ is the subset $[1, \infty)$.
The results applies for bilateral weighted shifts

\[ W e_n = w_n e_{n+1}, \quad n \in \mathbb{Z} \]

**Theorem**

Let us suppose that an operator \( T \) on a complex Banach space is similar to \( \alpha T \) for some complex number \( \alpha \). If \( \lambda \) is an extended eigenvalue for \( T \) then \( \lambda \alpha \) is an extended eigenvalue for \( T \).

**Corollary**

If \( W \) is a bilateral weighted shift then every \( \lambda \in \mathbb{T} \) is an extended eigenvalue for \( W \).

**Theorem**

If \( \lambda \) is an extended eigenvalue of a bilateral weighted shift \( W \) whose point spectrum has non-empty interior then \( |\lambda| = 1 \).
Question (Shields’1974)

Let $W$ be a invertible bilateral weighted shift. Is there exist a non-trivial closed subspace invariant for $W$ and $W^{-1}$? Is there exist a non trivial invariant subspace for $W + W^{-1}$?

Question

Which are the extended eigenvalues for a bilateral weighted shift and their corresponding extended eigenoperators?
Intertwining relations

Definition

A bounded operator $A$ intertwines with a bounded operator $B$ provided there exists a bounded operator $X \neq 0$ such that $AX = XB$.

Question

Let $A, B$ two bilateral weighted shifts. When $A$ intertwines with $B$?
Shields’74

An operator $X$ intertwines two bilateral weighted shifts $A$ and $B$ with sequences of weights $(\alpha_n)n \in \mathbb{Z}$ and $(\beta_n), n \in \mathbb{Z}$ if and only if

$$\beta_j x_{i+1,j+1} = \alpha_i x_{i,j}$$

where $x_{i,j} = \langle Xe_j, e_i \rangle$ are the coefficients of the matrix of $X$ with respect to the canonical basis on $\ell^2(\mathbb{Z})$.

Theorem

Let $A$ and $B$ be two injective bilateral weighted shifts with sequences of weights $(\alpha_n), n \in \mathbb{Z}$ and $(\beta_n), n \in \mathbb{Z}$. Then, $A$ intertwines with $B$, if and only if there exist $k \in \mathbb{Z}$ and a constant $M$ such that

$$\frac{|\alpha_k \cdots \alpha_{k+n-1}|}{|\beta_0 \cdots \beta_{n-1}|} \leq M \quad \text{and} \quad \frac{|\beta_1 \cdots \beta_{n-1}|}{|\alpha_{k-1} \cdots \alpha_{k-n}|} \leq M$$
Theorem
Let \( W \) be a injective bilateral weighted shift. Then, the set of extended eigenvalues for \( W \) has only one of the following pictures: \( \mathbb{C} \setminus \mathbb{D} \) or \( \mathbb{C} \setminus \{0\} \) or \( \overline{\mathbb{D}} \setminus \{0\} \), or \( \mathbb{T} \).

Theorem
Let \( A \) be an injective bilateral weighted shift and let \( \lambda \) extended eigenvalue, with \( |\lambda| \neq 1 \). Then every extended eigenoperator for \( A \) corresponding to \( \lambda \) is strictly lower triangular.

Theorem
Let \( A \) be an injective bilateral weighted shift and let \( \lambda \in \mathbb{T} \). Then every extended eigenoperator \( X \) for \( A \) corresponding to \( \lambda \) factors as \( X = D_\lambda B \) for some \( B \in \{A\}' \). (\( D_\lambda e_n = \lambda^{-n}e_n \)).
Questions

1. Show that if $X$ is an extended eigenoperator for $C_1$ on $L^p[0, 1]$ then there exists $R \in \{C_1\}'$ such that $X = X_0 R$, where $X_0$ is a fixed eigenoperator.

2. Show that if $1 < p < \infty$ and if $\lambda$ is real and $\lambda \geq 1$ then $\lambda$ is an extended eigenvalue for $C_0$ on $\ell^p$.

3. How we can weaken the conditions of intertwining in Brown and Kim-Mooore-Pearcy’s theorem on the special case of the bilateral weighted shift in order to obtain new results on hyperinvariant subspace for bilateral weighted shifts.


