Geometric properties of cones having a large dual

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(Joint work in progress with M. A. Melguizo Padial)

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XIII Encuentro de la Red de Análisis Funcional y Aplicaciones
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1 Introduction

2 Main result and consequences

3 Bibliography
Notation and terminology

- $X$ denotes a normed space

An open half space of $X$ is a set $\{x \in X : f(x) < \lambda\}$ for some $f \in X^* \setminus \{0\}$ and $\lambda \in \mathbb{R}$. We denote it briefly by $\{f < \lambda\}$.

An slice of a set $C$ is a non-empty intersection of $C$ with an open half space of $X$.

$$f = \lambda \quad C_S = \{f < \lambda\} \cap C$$
**Notation and terminology**

- $X$ denotes a normed space
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**Definition**

Let $C$ be a subset of $X$, $c \in C$ is said to be a **denting point of $C$** if

$$c \not\in \text{conv}(C \setminus B_\varepsilon(c)), \ \forall \varepsilon > 0.$$
Denting points

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Denting points

Dentability is applied to study
- Radon-Nikodým property
- LUR renorming
- Optimization
- Operators theory
Points of continuity

Definition

Let $C$ be a subset of $X$, $c \in C$ is said to be a point of continuity for $C$ if the identity map $(C, \text{weak}) \to (C, \|\|)$ is continuous at $c$. 
Points of continuity

$c$ is a point of continuity for $C$ if and only if for every open ball $B_\varepsilon(c)$, there exists a weakly open $U$ such that

$$c \in U \cap C \subset B_\varepsilon(c) \cap C$$
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The notion of point of continuity is applied to
- Provide a geometric proof a fixed point theorem
- Geometric properties related to Radon-Nikodým property
- Optimization
denting point $\Rightarrow$ point of continuity
denting point ⇒ point of continuity

**Definition**

$c$ is an extreme point of $C$ if it does not belong to any non degenerate line segment in $C$
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Denting points and points of continuity

denting point $\implies$ point of continuity

**Definition**

c is an extreme point of $C$ if it does not belong to any non degenerate line segment in $C$. 

![Extreme points](image-url)
**Theorem (Lin–Lin–Troyanski, 1985)**

Let $c$ be an extreme point of a closed, convex, and bounded subset $C$ of a Banach space. If $c$ is a point of continuity for $C$, then it is a denting point.
Denting points and points of continuity

**Theorem (Lin–Lin–Troyanski, 1985)**

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What about cones?
A non empty convex subset $C$ of $X$ is called a cone if

$$\alpha C \subseteq C, \forall \alpha \geq 0$$
Introduction

**Denting points, points of continuity, and cones**

**Definition**

1. A non empty convex subset $C$ of $X$ is called a cone if

   $$\alpha C \subset C, \quad \forall \alpha \geq 0$$

2. A cone $C$ is called **pointed** if $C \cap (-C) = \{0_X\}$

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Definition

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Problem (Gong, 1995)

The property of point of continuity at the origin for a closed and pointed cone in a normed space, is really weaker than the property of denting point at the origin of the cone?
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A negative answer for Banach spaces

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**Theorem (Daniilidis, 2000)**

Let $C$ be a closed and pointed cone in a Banach space $X$. Then $0_X$ is a denting point of $C$ if and only if it is a point of continuity for $C$. 
Denting points, points of continuity, and cones

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The property of point of continuity at the origin for a closed and pointed cone in a normed space, is really weaker than the property of denting point at the origin of the cone?

The former characterization allowed Daniilidis to prove the equivalence (into the frame of Banach spaces) between two density results of Arrow, Barankin and Blackwell’s type. One due to Petschke (1990) and another due to Gong (1995).
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Denting points, points of continuity, and cones

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A positive answer for non closed cones

Example (GC–Melguizo–Montesinos, 2015)
Let us define $X := \mathbb{R}^2$ and $C := \mathbb{R} \times (0, +\infty) \cup \{(0, 0)\}$ which is a pointed cone. Then $0_X$ is point of continuity for $C$ but it is not a denting point.
Problem (Gong, 1995)

The property of point of continuity at the origin for a closed and pointed cone in a normed space, is really weaker than the property of denting point at the origin of the cone?

The problem still remains open for closed cones
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Denting points, points of continuity, and cones

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The problem still remains open for closed cones

What assumption provides the equivalence?
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What assumption provides the equivalence?

\[ C^* := \{ f \in X^* : f(c) \geq 0, \forall c \in C \} \]
Denting points, points of continuity, and cones

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What assumption provides the equivalence?

\[ C^* := \{ f \in X^* : f(c) \geq 0, \ \forall c \in C \} \]

\[ [x, y] := \{ z \in X : x \leq z \leq y \} \]
Denting points, points of continuity, and cones

Problem (Gong, 1995)
The property of point of continuity at the origin for a closed and pointed cone in a normed space, is really weaker than the property of denting point at the origin of the cone?

Theorem (Kountzakis–Polyrakis, 2006)
Let $X$ be a normed space such that $\exists f \in C^*$ such that $X^* = \bigcup_{n \geq 1} [-nf, nf]$. Then $0_X$ is a denting point of a pointed cone $C$ if and only if it is a point of continuity for $C$. 
Problem (Gong, 1995)
The property of point of continuity at the origin for a closed and pointed cone in a normed space, is really weaker than the property of denting point at the origin of the cone?

Given $C \subset X \Rightarrow \tilde{C}$ denotes the closure of $C$ in $(X^{**}, \text{weak}^*)$

Theorem (GC-Melguizo-Montesinos, 2015)
Let $X$ be a normed space, $0_X$ is a denting point of a pointed cone $C$ if and only if it is a point of continuity for $C$ and $\tilde{C} \subset X^{**}$ is pointed.
**Theorem 1 (GC-Melguizo)**

Let $X$ be a normed space and $C \subset X$ a pointed cone. The following are equivalent:

(i) $0_X$ is a denting point of $C$.

(ii) There exist $n \in \mathbb{N}$, $\{f_i\}_{i=1}^{n} \subset C^*$, and $\{\lambda_i\}_{i=1}^{n} \subset (0, +\infty)$ such that the set, $\cap_{i=1}^{n} \{f_i < \lambda_i\} \cap C$, is bounded.

(iii) $0_X$ is a point of continuity for $C$ and $\overline{C^* - C^*} = X^*$ (i.e., $C^*$ is quasi-generating).

(iv) $\exists f \in C^*$ such that $X^* = \bigcup_{n\geq 1} [-nf, nf]$ (i.e., $C^*$ has an order unit).

(v) There exists $\{f_n\}_{n \geq 1} \subset C^*$ such that $X^* = \bigcup_{n \geq 1} [-nf_n, nf_n]$. 
Corollary 1 (GC-Melguizo)

Let $X$ be a normed space with a quasi-generating order cone $C \subset X$. If the origin is denting in $C$, then the following statements hold true:

(i) Every linear and positive operator $T : X^* \to X^*$ is continuous. In addition, if $T$ is not a multiple of the identity, then it has a nontrivial hyperinvariant subspace.

(ii) If a positive contraction $T : X^* \to X^*$ has 1 as an eigenvalue, then there exits an $0 < f \in X^{**}$ such that $T'f = f$. 
Corollary 2 (GC-Melguizo)

Let $X$ be a normed space and $C$ a pointed cone. If $0_X$ is a point of continuity for $C$ and $C^* \subset X^*$ is quasi-generating, then each weakly compact subset of $X$ has super efficient points.
**Proposition 1 (GC-Melguizo)**

Let $X$ be a normed space and $C \subset X$ a pointed cone. If $C$ is closed, then $C^* - C^*_{weak^*} = X^*$. 

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**Definition**

Let $X$ be a normed space and $C \subset X$ a cone. It is said that $0 \in X$ is a weakly strongly extreme point of $C$ if given two sequences $(c_n)$ and $(\tilde{c}_n)$ in $C$ such that $\lim_{n \to \infty} (c_n + \tilde{c}_n) = 0$, then $\lim_{n \to \infty} c_n = 0$. 

**Proposition 2 (GC-Melguizo)**

Let $X$ be a normed space and $C \subset X$ a pointed cone. If $0 \in X$ is a weakly strongly extreme point of $C$, then $C^* - C^* = X^*$. 
### Proposition 1 (GC-Melguizo)

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In general, closed cones do not have quasi-generating dual cones.
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Definition

Let $X$ be a normed space and $C \subset X$ a cone. It is said that $0_X$ is a weakly strongly extreme point of $C$ if given two sequences $(c_n)_n$ and $(\tilde{c}_n)_n$ in $C$ such that $\lim_n(c_n + \tilde{c}_n) = 0$, then weak−$\lim_n c_n = 0$. 
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Let $X$ be a normed space and $C \subset X$ a pointed cone. If $0_X$ is a weakly strongly extreme point of $C$, then $C^* = C^*$ weak$^*$ = $X^*$.
**Definition**

A cone $C$ in a normed space $X$ is said to be **normal** whenever $0 \leq x_n \leq y_n$ in $X$ and $\lim_{n} y_n = 0$ imply $\lim_{n} x_n = 0$. 

**Proposition 3 (GC-Melguizo)**

Let $X$ be a normed space and $C \subset X$ a pointed cone. If $C$ is normal, then $0_X$ is a weakly strongly extreme point of $C$.

**Corollary 3 (GC-Melguizo)**

Let $X$ be a normed space and $C \subset X$ a normal pointed cone. Then $0_X$ is a point of continuity for $C$ if and only if it is a denting point of $C$. 

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**Corollary 3 (GC-Melguizo)**

Let $X$ be a normed space and $C \subset X$ a normal pointed cone. Then $0_X$ is a point of continuity for $C$ if and only if it is a denting point of $C$. 
Definition

A norm $\| \|$ on a vector space $X$ is called to be strictly convex if given $x, y \in X$ with $\| x \| = \| y \| = 1$ and $\| x + y \| = 2$, we get $x = y$. 
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**Corollary 4 (GC-Melguizo)**

Let $X$ be a normed space and $C$ a pointed cone. If the norm of $X^{**}$ is strictly convex, then $\overline{C^* - C^*} = X^*$. As a consequence, $0_X$ is a point of continuity for $C$ if and only if it is a denting point of $C$. 
Example 1 (GC-Melguizo)

Let $\Gamma$ be an abstract nonempty set, consider the vector space

$$c_{00}(\Gamma) := \{(x_\gamma)_{\gamma \in \Gamma} \in l_\infty(\Gamma) : \{\gamma \in \Gamma : x_\gamma \neq 0\} \text{ is finite}\},$$

the non-complete normed space $(c_{00}(\Gamma), \| \cdot \|_\infty)$, where

$$\|(x_\gamma)_{\gamma \in \Gamma}\|_\infty := \sup\{|x_\gamma| : \gamma \in \Gamma\},$$

and the order cone

$$c_{00}(\Gamma)^+ := \{(x_\gamma)_{\gamma \in \Gamma} \in c_{00}(\Gamma) : x_\gamma \geq 0, \forall \gamma \in \Gamma\}.$$

Then the dual cone $(c_{00}(\Gamma)^+)^* \subset (c_{00}(\Gamma), \| \cdot \|_\infty)^*$ is quasi-generating and the origin is not a point of continuity for $c_{00}(\Gamma)^+$. 
Example 2 (GC-Melguizo)

Let us consider the non-complete normed space \((C_{00}(\mathbb{R}), \| \|_\infty)\), where \(\| f \|_\infty := \sup \{|f(x)| : x \in \mathbb{R}\}\) and the order cone

\[
C_{00}(\mathbb{R})^+ := \{ f \in C_{00}(\mathbb{R}) : f(x) \geq 0, \forall x \in \mathbb{R}\}.
\]

Then the dual cone \((C_{00}(\mathbb{R})^+)^* \subset (C_{00}(\mathbb{R}), \| \|_\infty)^*\) is quasi-generating and the origin is not a point of continuity for \(C_{00}(\mathbb{R})^+\).
Example 3 (GC-Melguizo)

Let us fix any $k \geq 1$, consider the vector space $C^k[a, b]$ of all functions on $[a, b]$ that have $k$ continuous derivatives, the non-complete normed space $(C^k[a, b], \| \|_\infty)$, where $\| f \|_\infty := \sup \{|f(x)| : x \in [a, b]\}$, and the order cone

$$C^k[a, b]^+ := \{ f \in C^k[a, b] : f(x) \geq 0, \forall x \in [a, b] \}.$$ 

Then the dual cone $(C^k[a, b]^+)^* \subset (C^k[a, b], \| \|_\infty)^*$ is quasi-generating and the origin is not a point of continuity for $C^k[a, b]^+$. 


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