Mathematics and Finance

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Mathematical Finance serves as a prime example of a flourishing application of mathematical theory. It became an important tool for several tasks in the financial industry and this development towards a "mathematization" of the financial business seems to be irreversible.

I want to briefly resume how these ideas developed, starting from the seminal work of Louis Bachelier who defended his thesis "Théorie de la spéculation" in 1900 in Paris. Henri Poincaré was a member of the jury and wrote a very positive report. Bachelier used probalistic arguments, thus introducing Brownian motion for the first time (five years before A. Einstein did so too). He used it as a mathematical model, in order to develop a theory of option pricing.

This theme subsequently remained dormant for almost 70 years until it was taken up again by the eminent economist Paul Samuelson. In the sequel Fisher Black, Robert Merton, and Myron Scholes applied a slightly modified version of Bachelier's model and the resulting "Black-Scholes formula" for the price of a European option quickly became very influential in the world of finance.

Bachelier was interested in designing a rational theory for the prices of term contracts. The two forms which were traded at the Bourse de Paris at that time also play a basic role today: forward contracts and options. We shall focus on the mathematically more interesting of these two derivates, namely options.

A European call (resp. put) option on an underlying security S consists in the right (but not the obligation) *to buy (resp. to sell) a fixed quantity of the underlying security S, at a fixed price K and a fixed time T in the future.*

The underlying security S, usually called the *stock*, can be a share of a company, a foreign currency, gold etc etc. In the case of Bachelier the underlying securities were *"rentes"*, a form of perpetual bonds which were very common in France in the nineteenth century.

Fixing the letter K for the strike price of the option, one arrives — after a moment's reflection — at the usual "hockey-stick" shape for the pay-off function of a call option at time T. We draw the value of the option as a function of the price S_T of the underlying asset S at time T.

Let C denote the price of the option.

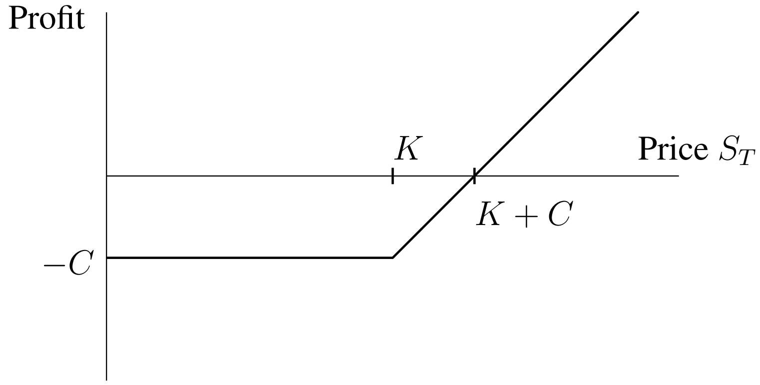

Figure 1: Pay-off function of a call option at time T.

The graph displayed in Figure 1 appears explicitly in Bachelier's thesis. It gives the profit or loss of the option at time $t = T$ when we shall know the price S_T of the underlying asset S. But we have to determine the price C of the option which we have to pay at time $t = 0$. This is our paradigmatic problem.

Louis Bachelier now passes to the central topic, *Probabilities in Operations on the Exchange*. Somewhat ironically, he had already addressed the basic difficulty of introducing probability in the context of the stock exchange in the introduction to the thesis in a very sceptical way: "The calculus of probabilities, doubtless, could never be applied to fluctuations in security quotations, and the dynamics of the Exchange will never be an exact science."

Nevertheless he now proceeds to model the price process of securities by a probability distribution distinguishing "two kinds of probabilities":

1. The probability which might be called "mathematical", which can be determined *a priori* and which is studied in games of chance.

2. The probability dependent on future events and, consequently impossible to predict in a mathematical manner.

This last is the probability that the speculator tries to predict."

My personal interpretation of this — somewhat confusing — definition is the following: sitting daily at the stock exchange and watching the movement of prices, Bachelier got the same impression that we get today when observing price movements in financial markets, e.g., on the internet. The development of the charts of prices of stocks, indices etc. on the screen or on the blackboard resembles a "game of chance". On the other hand, the second kind of probability seems to refer to the expectations of a speculator who has a personal opinion on the development of prices. Bachelier continues:

"His (the speculator's) inductions are absolutely personal, since his counterpart in a transaction necessarily has the opposite opinion."

This insight leads Bachelier to the remarkable conclusion, which in today's terminology is called the "efficient market hypothesis":

"It seems that the market, the aggregate of speculators, at a given instant can believe in neither a market rise nor a market fall since, for each quoted price, there are as many buyers as sellers."

He then makes clear that this principle should be understood in terms of "true prices", i.e., discounted prices. Finally he ends up with his famous dictum:

"In sum, the consideration of true prices permits the statement *of this fundamental* principle:

The mathematical expectation of the speculator is zero."

This is a truly fundamental principle and the reader's admiration for Bachelier's pathbreaking work will increase even more when continuing to the subsequent paragraph of Bachelier's thesis:

"It is necessary to evaluate the generality of this principle carefully: It means that the market, at a given instant, considers not only currently negotiable transactions, but even those which will be based on a subsequent fluctuation in prices as having a zero expectation.

For example, I buy a bond with the intention of selling it when it will have appreciated by 50 centimes. The expectation of this complex transaction is zero exactly as if I intended to sell my bond on the liquiditation date, or at any time whatever."

In my opinion, in these two paragraphs, the basic ideas underlying the concepts of martingales, stopping times, trading strategies, and Doob's optional sampling theorem already appear in a very intuitive way. It also sets the basic theme of the modern approach to option pricing which is based on the notion of a martingale.

The notion of a martingale describes in a mathematical way a totally fair game: in whichever way you place your bets on such a game your gain/loss equals in expectation zero. Bachelier's examples serve as a very intuitive interpretation: When having the chance to bet on a martingale, it is not possible to be smart, i.e. of creating positive expected values.

Bachelier applied his *fundamental principle* to the original problem of valuating an option and found a very interesting relation, i.e. an explicit way to derive its value from this basic principle.

We now make a big jump in time and turn to the seventies of the twentieth century when Bachelier's work experienced a renaissance through the work of Paul Samuelson, Fisher Black, Myron Scholes, and Robert Merton. These authors did not - a priori - believe in Bachelier's "Fundamental Principle". Rather they made the more modest assumption that the price process of a stock should not be totally unfair. To formalize this intuitive idea the notion of arbitrage appears: an arbitrage opportunity is a combination of bets which allows you to make a sure win. The "no arbitrage principle" postulates that it should not be possible to make an arbitrage when betting in a financial market.

Clearly the "no arbitrage" assumption on a game is much weaker than postulating that the game is a martingale. In other words: demanding that a game is not totally unfair (i.e. allowing an arbitrage) is much less than demanding that it is perfectly fair.

Nevertheless, based only on the much weaker assumption of "no arbitrage" the above mentioned authors derived the "Black-Scholes option pricing formula" which is analogous to Bachelier's results.

How do these two concepts fit together?

The connection between the notion of "no arbitrage" and the notion of a "martingale" is the theme of the *Fundamental Theorem of Asset Pricing*. This theorem states a remarkable dichotomy which applies to every conceivable game: there are only two possibilities. Either the game is totally unfair in the sense that it allows for an arbitrage. Ruling out this extreme case of unfairness we remain with the second possibility: in this case we can always "change the odds", i.e the probabilities of the possible events, so that after this change the game becomes perfectly fair, i.e., a martingale.

This theorem reconciles the approaches of Bachelier and Black-Scholes and paved the way for a spectacular development of the field of Mathematical Finance. It was first formulated in the late seventies by Michael Harrison, David Kreps, and Stan Pliska in a somewhat narrow framework. The mathematical challenge remained to formulate and prove the theorem in a general framework.

In 1992 I was lucky to join forces with Freddy Delbaen to tackle this problem. I knew Freddy for a long time from our joint interest in the field of functional analysis, a field where also my laudator, José Orihuela, is active. It turned out that the detailed knowledge of some key techniques from functional analysis was the key for cracking the hard nut of finding the correct statement and proof of this "Fundamental Theorem". In 1994 we published the solution which - according to Google Scholar - has been cited some 2.000 times since. This paper played a central role in the development of the field of mathematical finance in the past 25 years.

With this short survey on the "Fundamental Theorem" I wanted to sketch the underlying idea in a - hopefully - non-technical way to give an insight into our mathematical work.

It remains to sincerely thank the University of Murcia for granting me the eminent honor of a doctorate honoris causa.