On the monotonicity of the isoperimetric quotient for parallel bodies

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1 Monotonicity of the relative isoperimetric quotient of parallel bodies

Let $\mathcal{K}^n$ denote the family of all full-dimensional convex bodies in $\mathbb{R}^n$, and let $E \in \mathcal{K}^n$ be a fixed gauge body. The inradius of K w.r.t. E is $\mathrm{r}(K;E) = \max\{r \geq 0 : \text{ there is some } x \in \mathbb{R}^n \text{ such that } x + rE \subseteq K\}$. The parallel bodies of K w.r.t. E are

$$K_\lambda = \begin{cases} \{x \in \mathbb{R}^n : x + |\lambda| E \subseteq K\} & \text{for } -r(K;E) \leq \lambda \leq 0 \quad \text{(inner parallel bodies)}, \\ K + \lambda E & \text{for } \lambda \geq 0 \quad \text{(outer parallel bodies)}. \end{cases}$$

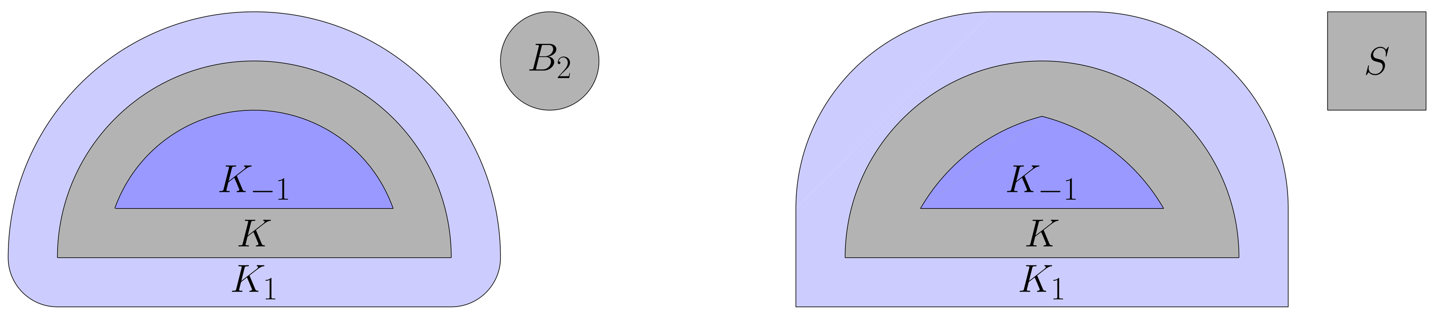

Figure 1: Inner and outer parallel bodies of a Euclidean semi-disc K relative to a Euclidean disc $E = B_2$ and to a square $E = S$.

The relative quermassintegrals $\mathrm{W}_i(K;E)$, $0 \leq i \leq n$, of K w.r.t. E are defined by the relative Steiner formula for the volume of the Minkowski sum $K + \lambda E$,

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\lambda^i, \qquad \lambda \geq 0.$$

In particular, $\mathrm{S}(K;E) = n\,\mathrm{W}_1(K;E)$ is the relative surface area (see e.g. [3, Section 5.1.2]). The relative isoperimetric quotient [5], a quantity that is invariant under homothetic scaling of K, is

$$\mathrm{I}(K;E) = \frac{\mathrm{S}(K;E)^n}{\mathrm{vol}(K)^{n-1}}.$$

The body K is a tangential body of $E \subseteq K$ if through every boundary point of K there is a supporting hyperplane of K that also supports E [8, p. 149].

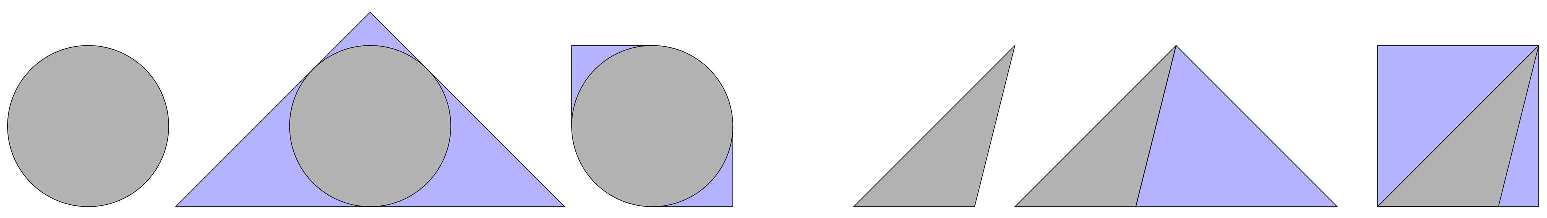

Figure 2: Some tangential bodies of a Euclidean disc (left) and of a triangle (right).

Main result

Theorem 1. *The relative isoperimetric quotient function*

$$\mathrm{I}(\lambda) = \frac{\mathrm{S}(K_\lambda;E)^n}{\mathrm{vol}(K_\lambda)^{n-1}}$$

is monotonically decreasing on $(-\mathrm{r}(K;E), \infty)$.

Moreover, the following are equivalent for all $-\mathrm{r}(K;E) < \lambda_0 < \lambda_1 < \infty$:
(i) $\mathrm{I}(\lambda_0) = \mathrm{I}(\lambda_1)$,
(ii) K_{λ_0} *is homothetic to* K_{λ_1},
(iii) K_{λ_1} *is homothetic to a tangential body of* E,
(iv) $\mathrm{I}(\lambda)$ *is constant on* $(-\mathrm{r}(K;E), \lambda_1]$.

If $\lambda_1 > 0$, the equivalent conditions (i)-(iv) are satisfied if and only if K is homothetic to E and, consequently, if and only if $\mathrm{I}(\lambda) = n^n \mathrm{vol}(E)$ for all $\lambda \in (-\mathrm{r}(K;E), \infty)$.

Remark 2. (a) *The monotonicity for $\lambda \geq 0$ is not new, see [5, Remark 4.4].*
(b) *When the gauge body E is the Euclidean unit ball B_n, we obtain the classical surface area $\mathrm{S}(K;B_n) = \mathrm{S}(K)$ and the isoperimetric quotient function amounts to $\mathrm{I}(\lambda) = \frac{\mathrm{S}(K_\lambda)^n}{\mathrm{vol}(K_\lambda)^{n-1}}$. Even in that central case the monotonicity for inner parallel bodies, i.e. for $\lambda < 0$, seems to be not present in the literature.*

2 Monotonocity for related families and isoperimetric problems

Let Ω be a subset of the Euclidean unit sphere $\mathbb{S}^{n-1}$ containing the origin in the interior of its convex hull. Let $K^\Omega = \bigcap_{u \in \Omega} H^-_{u,h_K(u)}$, whence $K \subseteq K^\Omega \in \mathcal{K}^n$. Then K is determined by Ω if $K^\Omega = K$ [8, pp. 385, 411].

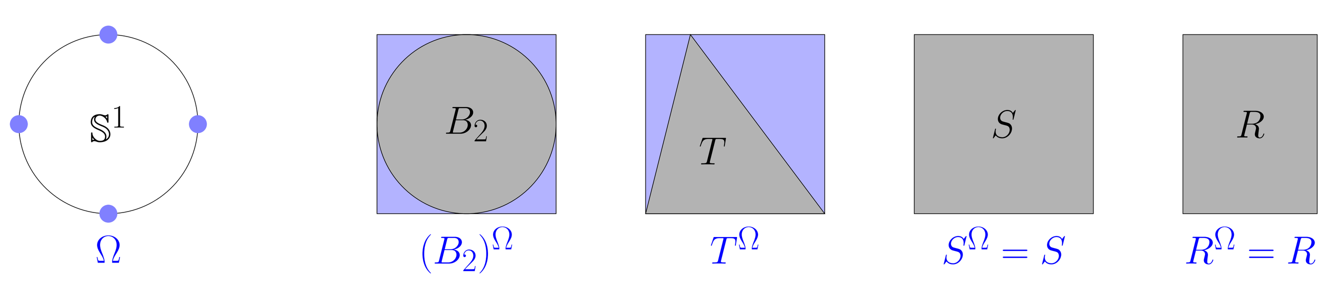

Figure 3: Bodies K^Ω for $\Omega = \{(\pm 1, 0), (0, \pm 1)\}$ and K being a Euclidean disc, a triangle, a square and a rectangle. The square and the rectangle are determined by Ω, whereas the disc and the triangle are not.

When Ω determines K, we define

$$K(\Omega, \lambda) = \bigcap_{u \in \Omega} H^-_{u,h_K(u)+\lambda h_E(u)}, \qquad \lambda \geq -\mathrm{r}(K;E).$$

This generalizes the family $(K_\lambda)_{\lambda \geq -\mathrm{r}(K;E)}$ of parallel bodies in so far as $K(\Omega,\lambda) = K_\lambda$ for $-\mathrm{r}(K;E) \leq \lambda \leq 0$ whenever Ω determines K and $K(\mathbb{S}^{n-1}, \lambda) = K_\lambda$ for all $\lambda \geq -\mathrm{r}(K;E)$.

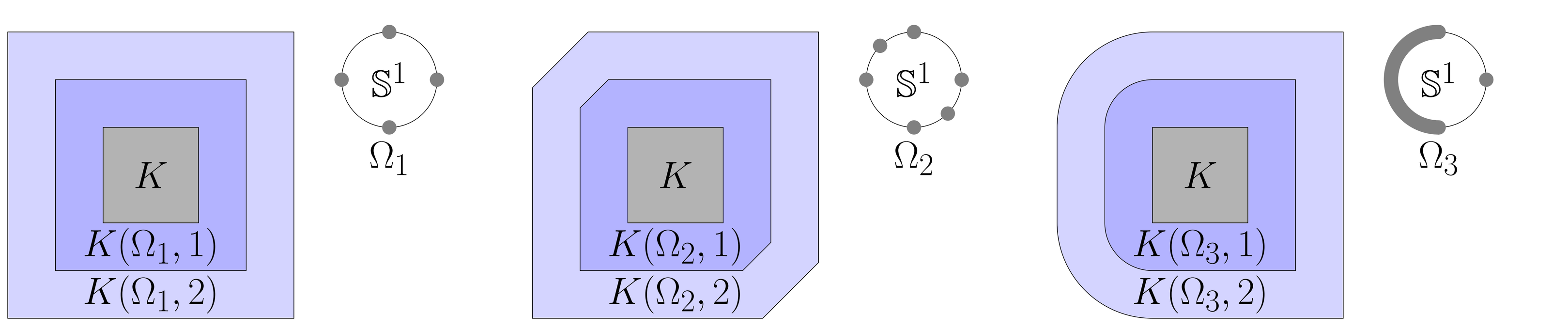

Figure 4: Examples of $K(\Omega, \lambda)$, $\lambda = 1, 2$, with K being a square, $E = B_2$ being a Euclidean disc and different choices of $\Omega \subseteq \mathbb{S}^1$.

Main results

Theorem 3. *If $\Omega \subseteq \mathbb{S}^{n-1}$ determines K, the relative isoperimetric quotient function*

$$\mathrm{I}^\Omega(\lambda) = \frac{\mathrm{S}(K(\Omega,\lambda);E)^n}{\mathrm{vol}(K(\Omega,\lambda))^{n-1}}$$

of the family $(K(\Omega,\lambda))_{\lambda > -\mathrm{r}(K;E)}$ is monotonically decreasing on $(-\mathrm{r}(K;E), \infty)$.

Moreover, the following are equivalent for all $-\mathrm{r}(K;E) < \lambda_0 < \lambda_1 < \infty$:
(i) $\mathrm{I}^\Omega(\lambda_0) = \mathrm{I}^\Omega(\lambda_1)$,
(ii) $K(\Omega,\lambda_0)$ *is homothetic to* $K(\Omega,\lambda_1)$,
(iii) $K(\Omega,\lambda_1)$ *is homothetic to a tangential body of* E,
(iv) $\mathrm{I}^\Omega(\lambda)$ *is constant on* $(-\mathrm{r}(K;E), \lambda_1]$.

If $\lambda_1 > 0$, the equivalent conditions (i)-(iv) are satisfied if and only if K is homothetic to E^Ω and, consequently, if and only if $\mathrm{I}(\lambda) = \mathrm{I}\left(E^\Omega; E\right) = n^n \mathrm{vol}\left(E^\Omega\right)$ for all $\lambda \in (-\mathrm{r}(K;E), \infty)$

The shape of $K(\Omega, \lambda)$ tends to E^Ω (up to homothety) as $\lambda \to \infty$. This yields the following.

Corollary 4. *Let $\Omega \subseteq \mathbb{S}^{n-1}$ be a set that contains the origin in the interior of its convex hull. Then a convex body $K_0 \in \mathcal{K}^n$ is a minimizer of the relative isoperimetric quotient $\mathrm{I}(\cdot;E)$ among all convex bodies $K \in \mathcal{K}^n$ that are determined by Ω if and only if K_0 is homothetic to the tangential body E^Ω of E. In particular, that minimal quotient is $\mathrm{I}\left(E^\Omega; E\right) = n^n \mathrm{vol}\left(E^\Omega\right)$.*

Remark 5. *If $E = B_n$, Corollary 4 concerns the classical isoperimetric quotient as mentioned in Remark 2(b). For that case the claim is a well-known result [8, p. 385], that goes back to Lindelöf and Minkowski [6, 7] for finite Ω and to Aleksandrov [1] for general Ω.*
For $E = B_n$ and $\Omega = \mathbb{S}^{n-1}$, we obtain the isoperimetric inequality for arbitrary convex bodies.

3 Isoperimetric-type quotients of quermassintegrals

Here we ask for the monotonicity of all quotients

$$\frac{\mathrm{W}_j(K_\lambda;E)^{n-i}}{\mathrm{W}_i(K_\lambda;E)^{n-j}}, \qquad 0 \leq i < j < n.$$

Note that Theorem 1 covers the particular case $\dfrac{\mathrm{W}_1(K_\lambda;E)^n}{\mathrm{W}_0(K_\lambda;E)^{n-1}} = \dfrac{1}{n^n}\dfrac{\mathrm{S}(K_\lambda;E)^n}{\mathrm{vol}(K_\lambda)^{n-1}}$.

The body $K \in \mathcal{K}^n$ belongs to the class $\mathcal{R}_j$, $0 \leq j \leq n-1$, if

$$\frac{\mathrm{d}}{\mathrm{d}\,\lambda}\mathrm{W}_i(K_\lambda;E) = (n-i)\,\mathrm{W}_{i+1}(K_\lambda;E)$$

for all $0 \leq i \leq j$ and $-\mathrm{r}(K;E) \leq \lambda < \infty$. Note that $\mathcal{R}_{n-1} \subseteq \mathcal{R}_{n-2} \subseteq \ldots \subseteq \mathcal{R}_1 \subseteq \mathcal{R}_0 = \mathcal{K}^n$, and the inclusions are strict in general [4].

Main results

Theorem 6. *Let $0 \leq i < j < n$, and suppose that $K \in \mathcal{R}_{j-1}$. Then the function*

$$\mathrm{I}_{i,j}(\lambda) = \frac{\mathrm{W}_j(K_\lambda;E)^{n-i}}{\mathrm{W}_i(K_\lambda;E)^{n-j}}$$

is monotonically decreasing on $(-\mathrm{r}(K;E), \infty)$.

Moreover, if E is smooth, the following are equivalent for all $-\mathrm{r}(K;E) < \lambda_0 < \lambda_1 < \infty$:
(i) $\mathrm{I}_{i,j}(\lambda_0) = \mathrm{I}_{i,j}(\lambda_1)$,
(ii) K_{λ_0} *is homothetic to* K_{λ_1},
(iii) K_{λ_1} *is homothetic to an $(n-j)$-tangential body of E (cf. [8, p. 86]),*
(iv) $\mathrm{I}_{i,j}(\lambda)$ *is constant on* $(-\mathrm{r}(K;E), \lambda_1]$.

If E is smooth and $\lambda_1 > 0$, conditions (i)-(iv) are satisfied if and only if K is homothetic to E and, consequently, if and only if $\mathrm{I}_{i,j}(\lambda) = \mathrm{vol}(E)^{j-i}$ for all $\lambda \in (-\mathrm{r}(K;E), \infty)$.

The above monotonicity is shown in [5] under the stronger assumption that $K \in \mathcal{R}_j$.

Corollary 7. *Let $0 \leq i < j < n$ and let E be smooth. Then a convex body $K_0 \in \mathcal{R}_{j-1}$ is a minimizer of the quotient $\dfrac{\mathrm{W}_j(\cdot;E)^{n-i}}{\mathrm{W}_i(\cdot;E)^{n-j}}$ among all convex bodies $K \in \mathcal{R}_{j-1}$ if and only if K_0 is homothetic to E.*

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