WELFARE LOSSES UNDER OLIGOPOLY*

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Abstract

We find that welfare losses due to oligoplistic output setting in the Cournot model are small in a number of situations with as few as three competitors. However, welfare losses due to the wrong number of firms operating under oligopoly are found to be quite substantial.

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1. Introduction

In his classical contribution Cournot (1837, Chapter 8) established that when the number of firms in a market tends to infinity, oligopoly equilibrium tends to perfect competition. As a corollary, Welfare Losses (WL), measured as the difference between social welfare in the optimal and the equilibrium allocation, tend to zero. Our paper focuses attention on WL when the number of firms is small, i.e., under oligopoly. To the best of our knowledge there are only a handful of theoretical papers dealing with this problem from the classical work of Hotelling (1938), see McHardy (2000) and the references there. It seems to us that the question of the size of WL under oligopoly is crucial for two reasons: First, were WL small there would be little to be gained by considering oligopolistic behavior: A simple equilibrium concept like perfect competition may be preferable. Second, the motive for public policies would be weakened under small WL.

In Section 2 we will present the Cournot model of oligopoly with a given number of identical firms. Demand and costs are linear. The model allows to calculate the percentage of WL under oligopoly, denoted by PWL, as follows $PWL = 1/(1 + n)^2$ where $n$ is the number of firms. Thus a market composed by 7 firms would produce a PWL of 1.56%, not a big number. In order to appraise the size of WL under oligopoly and regulation we compare PWL in oligopoly with PWL under price regulation with unknown costs. The latter is chosen because it is a realistic alternative to "free market". We find that PWL under price regulation are larger under oligopoly except, possibly, in the duopoly case. Thus, the results in this section point out to the fact that oligopoly WL are small, an unpleasant result as we argued before. Thus, in the rest of the paper we consider the robustness of this result by considering the following scenarios:

1) More general form of the demand function (Section 3).
2) Non constant returns to scale (Section 4).

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A number of papers have studied the conditions under which this result occurs in general in large economies, Dasgupta and Ushio (1981), Fraysse and Moreaux (1981) and Guesnerie and Hart (1985). On the contrary, there is a sizeable literature measuring WL in actual economies following the work of Harberger (1954) (see Tullock (2003) p. 2 for a list of papers in this literature).

Other attempts to find higher WL focus on different issues: "X-Inefficiency" (a forerunner of moral hazard), see Leibenstein (1966) and Rent-Seeking (a forerunner of contest theory), see Tullock (1967).
3) Fixed Costs and free entry (Section 5).
4) Non identical firms (Section 6).

The consideration of Point 1 does not bring a great improvement in the values of PWL. However we find large WL associated with overentry under increasing returns or free entry. In these two cases, except for the case of two, or possibly three firms, WL due to the wrong output are also small.

Finally, in Section 7 we offer some thoughts about these results. Our main conclusion is that the search for WL under oligopoly should not focus on output misallocation, as it has been done so far, but to the wrong number of firms in the market.

It goes without saying that our main findings should be taken with some caution before a more complete picture of WL is available. For instance, it would be interesting to compare oligopolistic WL with those arising under other kinds of regulation like optimal regulation. Also important causes of WL are not considered here, i.e. product differentiation. Intuition suggest that this is, potentially, an important cause of WL because, in limiting cases, i.e. when products are almost independent, we are essentially back to the case of monopoly where WL can be substantial. Also we leave for future research the study of the impact of variables like investment, R&D, quality, location, etc., on WL. The analysis of the impact of these variables on WL requires the consideration of games that are far more complicated than those considered here.

2. The Linear Model

We will begin by considering the simplest model of oligopoly (see Yarrow (1985) for a similar approach). Suppose that there is a representative consumer with a utility function $U = Ax - \frac{bx^2}{2} - px$ where $x$ is aggregate output, $p$ is the market price and $A$ and $b$ are positive numbers. The maximization of utility generates an inverse demand function $p = A - bx$. There are $n$ identical firms each producing a single output denoted by $x_i$. Thus $x = \sum_{i=1}^{n} x_i$. Marginal cost is constant and denoted by $c$. Profits for firm $i$ are $\pi_i = (p - c)x_i$. Defining $a \equiv A - c$ we have that $\pi_i \equiv (a - bx)x_i$. Assume $a > 0$. If firms compete à la Cournot, first order condition of profit maximization yields $a = bx + bx_i$. It is easy to check that the second order condition holds and that
equilibrium is symmetric. Thus
\[ x^* = \frac{an}{b(n+1)}. \] (2.1)
Social welfare, denoted by \( W \), is the sum of profits and the utility of the consumer,
\[ W \equiv (p - c)x + Ax - \frac{bx^2}{2} - px = ax - \frac{bx^2}{2}. \] Thus, in a Cournot equilibrium,
\[ W^* = \frac{a^2 n(n + 2)}{2b(n + 1)^2}. \] (2.2)
In an optimal allocation, social welfare is maximized. First order condition yields
\[ x^o = \frac{a}{b}. \] (2.3)
Again, second order condition holds and the allocation is symmetric. Thus,
\[ W^o = \frac{a^2}{2b}. \] (2.4)
Define now the percentage of welfare loss (PWL) as follows
\[ PWL \equiv \frac{W^o - W^*}{W^0} = 1 - \frac{n(n + 2)}{(n + 1)^2} = \frac{1}{(n + 1)^2}. \] (2.5)
For the case of monopoly, \( PWL \) is undoubtedly large, .25. But \( PWL \) decreases sharply
with the number of competitors; under duopoly is .11, with three firms is .0625, for four firms, \( PWL \) is .04 and for five firms \( PWL \) is .027.\(^4\) A natural question is, are these
numbers large to justify regulation? In order to get an idea about that we compare these
WL with those arising under regulation when the regulator is not completely informed
about the characteristics of firms. If the latter are larger, oligopoly produces small WL
relative to those produced by regulation. For simplicity we will say that in this case,
oligopoly WL are small.

Suppose that the regulator controls the market price.\(^5\) She is uncertain of the value
of the marginal cost which can take two values: \( y \) with probability \( \pi (> 0) \) and \( z (< y) \)
\(^4\)This result shows that once linearity is assumed, WL never comes up to big numbers except if the
number of firms is very small. This may help to explain the small WL found by the empirical studies.
\(^5\)For the optimal regulation under oligopoly, see López-Cuñat (1995) who generalizes the analysis of
Baron and Myerson (1982) in the monopoly case. We consider here a kind of regulation that is simpler
and more in tune with real life practices.
with probability $1 - \pi$ ($> 0$). Once $p$ is set, firms decide output. If $p$ is smaller than the actual value of the marginal cost firms do to produce. In this case social welfare is zero. If $p$ is larger or equal than the actual value of the marginal cost, they produce an identical quantity each that satisfies demand $x = \frac{A-p}{b}$. In this case social welfare is $Ax - \frac{bx^2}{2} - cx = \frac{A-p}{b}(\frac{A+p}{2} - c)$ where $c = y, z$.

First we will show that a regulator interested in expected social welfare chooses either $p = y$ or $p = z$. Suppose that $p > y$. In this case expected social welfare, denoted by $EW$ is $\pi \frac{A-p}{b}(\frac{A+p}{2} - y) + (1 - \pi) \frac{A-p}{b}(\frac{A+p}{2} - z)$. Thus

$$\frac{\partial EW}{\partial p} = \frac{1}{b}[\pi(y-p) + (1 - \pi)(z-p)] = \frac{1}{b}(\pi y + (1 - \pi)z - p) < 0,$$

hence such a price can not be optimal. Suppose now that $y > p > z$. In this case expected social welfare is $EW = (1 - \pi) \frac{A-p}{b}(\frac{A+p}{2} - z)$. Thus

$$\frac{\partial E(p)}{\partial p} = (1 - \pi) \frac{-p + z}{b} < 0.$$

Again we find that a price decrease improves expected social welfare. Thus the price chosen by the regulator depends on the difference between $EW$ when $p = y$ ($\pi (\frac{A-y}{b})^2 + (1 - \pi) (\frac{A-y}{b}) (\frac{A+y-2z}{b})$) and $EW$ when $p = z$ ($(1 - \pi) (\frac{A-z}{b})^2$), i.e.

$$p = y \Leftrightarrow \pi \frac{(A-y)^2}{b} + (1 - \pi) \frac{(A-y)(A+y-2z)}{b} > (1 - \pi) \frac{(A-z)^2}{b} \tag{2.6}$$

Define: $a_H \equiv A - z$, $a_L \equiv A - y$ and $\beta \equiv \frac{a_H}{a_L}$. From (2.6) above, $p = y \Leftrightarrow (1 - \pi) \beta^2 - 2\beta(1 - \pi) + 1 - 2\pi < 0$. There is only one value of $\beta > 1$ for which this inequality is in fact a equality, namely $1 + \sqrt{\frac{\pi}{1-\pi}}$. It is easily seen that for larger values of $\beta$ the left hand side of the previous inequality is always positive. Thus, we can characterize the optimal policy in terms of $\pi$ and $\beta$, namely

$$p = y \Leftrightarrow \beta < 1 + \sqrt{\frac{\pi}{1-\pi}}, \quad p = z \Leftrightarrow \beta > 1 + \sqrt{\frac{\pi}{1-\pi}} \tag{2.7}$$

Consider first that $p = y$. Then, PWL of price regulation, denoted by $PWL^{PR}$ is

$$PWL^{PR} = \frac{\pi a_L^2}{\pi a_L^2 + (1 - \pi) \beta^2 a_L^2} = \frac{\pi}{\pi + (1 - \pi) \beta^2} \tag{2.8}$$
If the regulator achieves larger welfare than oligopoly, \( PWL^{PR} < PWL \). From (2.8) this is equivalent to \( \beta^2 > \frac{(1-PWL)\pi}{(1-\pi)PWL} \). From the latter inequality and (2.7), large WL under oligopoly require in this case that

\[
\beta < 1 + \sqrt{\frac{\pi}{1-\pi}} \quad \text{and} \quad \beta > \sqrt{\frac{(1-PWL)\pi}{(1-\pi)PWL}}
\]  

(2.9)

For the duopoly case, \( PWL = 1/9 \) so the second inequality in (2.9) reads \( \beta > 2.83\sqrt{\frac{\pi}{1-\pi}} \).

Both inequalities imply that \( \pi < .22897 \). For \( n = 3 \), \( PWL = 1/16 \) so the second inequality in (2.9) reads \( \beta > 3.87\sqrt{\frac{\pi}{1-\pi}} \). Both inequalities imply \( \pi < .10821 \) which looks like a small number. For larger number of firms \( PWL \) is smaller so the upper bound of \( \pi \) will be even smaller and so the likelihood of occurrence of large WL under oligopoly.

Let us now turn to the case where \( p = z \). Here PWL under price regulation is

\[
PWL^{PR} = (1-\pi)\frac{(a_H - a_L)^2}{\pi a_L^2 + (1-\pi)a_H^2} = \frac{(1-\pi)\beta^2}{\pi + (1-\pi)\beta^2}.
\]  

(2.10)

Again, if the regulator can achieve larger welfare than under oligopoly, \( PWL^{PR} < PWL \). From (2.10) this is equivalent to \( \beta^2 < \frac{PWLP\pi}{(1-\pi)(1-PWL)} \). From the latter inequality and (2.7), large WL under oligopoly require in this case that

\[
\beta > 1 + \sqrt{\frac{\pi}{1-\pi}} \quad \text{and} \quad \beta < \sqrt{\frac{PWLP\pi}{(1-\pi)(1-PWL)}}.
\]  

(2.11)

It is easy to see that for \( PWL < .5 \) there are no values of \( (\pi, \beta) \) solving the above inequalities. Thus, in this case WL of oligopoly are never large.

Summing up, except, possibly, for the case of duopoly, WL of oligopoly are small in the linear model compared with those arising under price regulation.

3. More General Demand Function

In this section we assume that the utility function of the representative consumer is \( U = Ax - \frac{bx^\alpha + 1}{\alpha + 1} - px \) with \( A \geq 0, b\alpha > 0 \) and \( \alpha > -1 \). The inverse demand function reads \( p = A - bx^\alpha \). Notice that if \( \alpha < 0 \) and \( A = 0 \) we have an isoelastic function
\( p = -bx^\alpha \) with \( b < 0 \) and \( \alpha < 0 \). The linear case considered in the Introduction occurs if \( \alpha = 1 \). First order condition of profit maximization yields \( a = bx^\alpha + bax^{\alpha-1}x_i \). Again it is easy to check that the second order condition holds. Thus Cournot equilibrium output is

\[
x^* = \left( \frac{an}{b(n+\alpha)} \right)^{\frac{1}{\alpha}}.
\]  

(3.1)

Social welfare is now \( W = ax - \frac{b x^{\alpha+1}}{1+\alpha} \) and the optimal aggregate output is

\[
x^o = \left( \frac{a}{b} \right)^{\frac{1}{\alpha}}.
\]  

(3.2)

Social welfare in equilibrium and in the optimal allocation, are, respectively

\[
W^* = \frac{a^{\frac{\alpha+1}{\alpha}} n^{\frac{1}{\alpha}} \alpha(n + \alpha + 1)}{b^{\frac{1}{\alpha}} (n + \alpha)^{\frac{1}{\alpha}} (n + \alpha)(\alpha + 1)} \quad \text{and} \quad W^o = \frac{a^{\frac{\alpha+1}{\alpha}} \alpha}{b^{\frac{1}{\alpha}} (\alpha + 1)}.
\]  

(3.3)

From there it follows that the percentage of welfare loss is

\[
PWL = 1 - \frac{n^{\frac{1}{\alpha}} (n + \alpha + 1)}{(n + \alpha)^{\frac{1}{\alpha} + \frac{1}{\alpha}}} \equiv PWL(n, \alpha).
\]  

(3.4)

How large this magnitude can be? It is easily shown that \( PWL \) decreases with \( n \). Also, since \( PWL \) can be written as

\[
1 - \frac{1 + \alpha/n + 1/n}{(1 + \alpha/n)^{\frac{1}{\alpha}} + \frac{1}{\alpha}}.
\]

It is readily seen that when \( n \to \infty \), \( PWL \to 0 \) and that when \( \alpha \to -1 \), \( PWL \to 0 \), regardless on \( n \). In order to study the function for other values of \( \alpha \) let us first compute the following:

\[
\frac{\partial PWL}{\partial \alpha} = \frac{n^{\frac{1}{\alpha}} (n + \alpha + 1)}{\alpha^2 (n + \alpha)^{\frac{1+\alpha}{\alpha}}} (\ln n - \ln (n + \alpha)) - \frac{n^{\frac{1}{\alpha}}}{(n + \alpha)^{\frac{1}{\alpha} + \frac{1}{\alpha}}} \left( 1 - \frac{1 + \alpha}{\alpha(n + \alpha)} \right).
\]

We see that at any point where \( \frac{\partial PWL}{\partial \alpha} = 0 \), the previous expression can be simplified by multiplying by \( \frac{\alpha^{\frac{\alpha+1}{\alpha}}}{n^{\frac{1}{\alpha}} (n + \alpha + 1)} \). Then

\[
0 = \ln (n + \alpha) - \ln n - \frac{\alpha(1 + n + 2\alpha)}{n^2 + 2a\alpha + a^2 + n + \alpha} = F(\alpha), \text{ say}.
\]  

(3.5)
Notice that \( \alpha = 0 \) is a solution of (3.5) but is not a solution to \( \frac{\partial PWL}{\partial \alpha} = 0 \) because we have multiplied the latter by \( \alpha^2 \). However any other solution to (3.5) is a solution to \( \frac{\partial PWL}{\partial \alpha} = 0 \). Now compute,

\[
\frac{\partial F(\alpha)}{\partial \alpha} = \frac{\alpha (\alpha^2 + \alpha - n^2 + 1)}{(n^2 + 2n\alpha + \alpha^2 + n + \alpha)^2} \tag{3.6}
\]

Notice the following facts:

1) There are only two values of \( \alpha \) for which \( \frac{\partial F(\alpha)}{\partial \alpha} = 0 \), namely \( \alpha = 0 \) and \( \alpha = \frac{-1 + \sqrt{4n^2 - 3}}{2} > 0 \).

2) For \( \alpha \approx -1 \), from (3.6) above, \( \frac{\partial F(\alpha)}{\partial \alpha} \approx \frac{1-n^2}{(n^2-n)^2} < 0 \).

3) For large values of \( \alpha \), \( \frac{\partial F(\alpha)}{\partial \alpha} > 0 \).

4) Finally, compute \( \frac{\partial^2 F(\alpha)}{\partial \alpha^2} \) which amounts to

\[
\frac{n + n^2 - 3\alpha^2 - \alpha + 6\alpha^2 n^2 + 2\alpha^3 n + 3\alpha^2 n + 3\alpha n^2 + 2n^3 \alpha - \alpha^4 - \alpha^3 - n^4 - n^3}{(n^2 + 2n\alpha + \alpha^2 + n + \alpha)^3}.
\]

From the expression above, we get that \( \frac{\partial^2 F(\alpha)}{\partial \alpha^2} = \frac{n + n^2 - n^4 - n^3}{(n^2 + n)^3} < 0 \).

Points 1-4 above imply that \( \frac{\partial F(\alpha)}{\partial \alpha} \) looks like in Figure 1 below (the figure is done for the case \( n = 2 \) but the shape is valid for any \( n \)):
Now $F(0) = 0$, $F(n)_{n \to \infty} \to \infty$, plus the shape of $\frac{\partial F(\alpha)}{\partial \alpha}$ obtained before, imply that $F(\alpha)$ looks like in Figure 2 below (the figure is done for the case of $n = 2$, but the shape is valid for any $n$).

\[ F(\alpha) = \ln (2 + \alpha) - \ln 2 - \frac{\alpha(3+2\alpha)}{6+5\alpha+\alpha^2} \]

FIGURE 2

Notice that the fact that $\alpha = 0$ is not a solution to $\frac{\partial PWL}{\partial \alpha} = 0$, and the shape of $F(\alpha)$ obtained before imply that there is only one value of $\alpha$ for which $\frac{\partial PWL}{\partial \alpha} = 0$.\footnote{Notice that PWL is not the integral of $F(\alpha)$ even though $\frac{\partial PWL(\alpha')}{\partial \alpha} = 0$ implies $F(\alpha') = 0$. This is due to the fact that we multiplied $\frac{\partial PWL}{\partial \alpha}$ by $\frac{\alpha^{(n+\alpha)}\ln \alpha}{\alpha \frac{\alpha}{\pi} (n+\alpha+1)}$ in order to obtain $F(\alpha)$.} Next, using the Maple calculator of Scientific Work we will find maximum values of $PWL$.

For $n = 2$ the maximum is almost .118, see Figure 3 below, only 6.3% higher than the .11 obtained in the linear case.\footnote{Using similar methods it can be shown that in the case of monopoly, the maximum PWL is, approximately, .264, only 5.8% higher than the .25 obtained under linear demand.} Notice that $F(\alpha)$ cannot be increasing for values of $\alpha$ larger than 30 because this would imply the existence of another point, say $\alpha$, $\frac{\partial PWL}{\partial \alpha} = 0 = F(\bar{\alpha})$ which we saw before it was impossible so indeed the maximum in the picture is a global maximum.
\[ PWL(2, \alpha) = 1 - 2^{1\frac{3+\alpha}{(2+\alpha)^{\frac{1+\alpha}{\alpha}}}} \]

When \( n = 3 \) the maximum is about .076 -see Figure 4 below- which is now 21% higher than the .0625 obtained in the case of linear demand.

\[ PWL(3, \alpha) = 1 - 3^{1\frac{4+\alpha}{(3+\alpha)^{\frac{1+\alpha}{\alpha}}}} \]

When \( n = 4 \) is approximately .056 which is 37.5% higher than the .04 obtained in the linear case, see Figure 5 below.
For \( n = 5 \) \( PWL \) is about .044, which is 58.8\% higher than the .0277 obtained in the linear case, see Figure 6 below.

Table 1 below shows the maximum \( PWL \) obtained in this section, denoted by \( PWL \), and the values obtained in the linear model, denoted by \( PWL \), for some values of \( n \): \( n \) between 2 and 10 capture most oligopolistic markets. When \( n = 20 \), many economists would say that the market is very close to perfect competition.
\[
\begin{array}{cccccccccccc}
 n & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 20 \\
 \text{PWL} & .118 & .076 & .056 & .044 & .0357 & .032 & .027 & .024 & .022 & .012 \\
 \text{PWL} & .11 & .0625 & .04 & .027 & .02 & .0156 & .012 & .01 & .008 & .003 \\
\end{array}
\]

\textbf{TABLE 1}

Notice that the relative difference between PWL and PWL increases with \( n \) and achieves very large numbers (for \( n = 10, 250\% \)). However this effect occurs for small values of PWL and is not strong enough to obtain large numbers. In the cases which we identify in the previous section as likely to yield WL, the maximum WL here and the WL there are substantially identical. Given this and that in the case considered here, WL can be much smaller than PWL, we have to conclude that the consideration of a more general class of demand functions does not bring significant WL associated with oligopoly.\(^8\)

4. Non Constant Returns to Scale

In this section we assume that the cost function is \( cx_i + x_i^2d/2 \). If \( d > 0 \) we have decreasing returns and if \( d < 0 \) we have increasing returns. In the latter costs become negative for sufficiently large output. We assume that \( 2c(d+b) > -da \) which guarantees that costs are positive in the optimum and in equilibrium and implies \( b > -d \). The latter guarantees that second order conditions of profit and welfare maximization hold. Now \( \pi_i = (a - bx - dx_i/2)x_i \) and \( W = ax - bx^2/2 - d/2 \sum_{i=1}^{n} x_i^2 \). First order condition of profit maximization is \( a - bx - (b + d)x_i = 0 \). From there, it is clear if, say, \( k \) firms are active, each produces an identical output \( x_i = \frac{a}{b + b(k+1)} \). Also, marginal profits evaluated at zero are \( \frac{a(b+d)}{b(k+1)+d} > 0 \), so in equilibrium all firms are active. Thus equilibrium is unique and it is

\(^8\)McHardy (2000) studies a model with quadratic demand and presents numerical calculations for several values of the parameters. He finds that in the case of duopoly WL can increase with respect to the linear case as much as 33%. This result differ from our’s because in our case the maximum increase is small (5.6%). In another case reported there, \( n = 5 \) and WL can increase as much as 30%, which is in tune with our results.
given by

\[ x^* = \frac{an}{d + bn(n + 1)}. \tag{4.1} \]

Similar arguments establish that the optimum is unique and is given by

\[ x^o = \frac{an}{d + bn} \text{ if } d > 0 \quad \text{and} \quad x^o = \frac{a}{d + b} \text{ if } d < 0. \tag{4.2} \]

From there we calculate that if \( d > 0 \)

\[ W^* = \frac{a^2n}{2(d + bn + b)} \quad \text{and} \quad W^o = \frac{a^2n}{2(bn + d)}. \]

\[ PWL = \frac{b^2}{(d + b(n + 1))^2} = \frac{1}{(d/b + n + 1)^2}. \tag{4.3} \]

Notice that WL are smaller than in the linear case. However if \( d < 0 \)

\[ W^o = \frac{a^2}{2(d + b)}. \]

From this expression and \( W^* \) we find that

\[ PWL = \frac{d^2 - dbn + 2db + b^2 - nd^2 - dbn^2}{(d + bn + b)^2}. \]

It is easy to see that for \( n > 1 \), \( PWL \) is decreasing on \( d \). Thus maximal \( PWL \), denoted by \( PWL \) obtains when \( d \) attains its minimum value.\(^9\) Since \( 2c(d + b) > -da \), a lower bound for \( d \) obtains when \( -d = \frac{2cb}{a + 2c} \). Thus, the maximal \( PWL \) is

\[ PWL = \frac{2cna + a^2 + 2cn^2a + 4c^2n^2}{(na + 2cn + a)^2}. \]

We see that when \( n \to \infty \),

\[ PWL \to \frac{4c^2 + 2ca}{a^2 + 4c^2 + 4ac} = \frac{2c}{a + 2c} = \frac{2c}{A + c}. \]

In this case we need no comparison with other resource allocation mechanisms to state that \( PWL \) can be quite substantive: For instance for \( 2c = A, PWL \to 2/3. \)

\(^9\)Minimum \( PWL \) occurs for the maximum value of \( d \), namely, \( d = 0 \). In this case \( PWL \) equals the value obtained in (1.5).
A natural question is: Are these WL due to the wrong output being chosen by oligopolists or to the fact that oligopoly entails too many firms? In order to disentangle these two effects let us maximize social welfare with the constraint that all firms produce the same output. Call the solution Second Best. Second best output is identical to the case \( d > 0 \) considered before and WL are given by (4.3) above. Again the maximum value of \( PWL \) in the second best, denoted by \( PWL_{SB} \), is obtained when \( d = \frac{-2b}{a+2c} \), which amounts to

\[
PWL_{SB} = \left( \frac{1}{n + \frac{a}{a+2c}} \right)^2.
\]

Clearly when \( n \to \infty \) \( PWL_{SB} \to 0 \) so when the number of firms is large there will be a huge discrepancy between total WL and WL due to the wrong output. Also we see that

\[
PWL_{SB} \in \left[ \frac{1}{(n+1)^2}, \frac{1}{n^2} \right].
\]

Notice that the upper (resp. lower) bound of \( MPWL_{SB} \) is not far, especially for \( n > 3 \) (resp. identical) to the expression obtained under constant returns.

Summing up, increasing returns may bring quite substantial WL. However, WL arising from output misallocation are not large.\(^{10}\) It is the fact that all firms except one are redundant from the social point of view what makes WL significant.

5. Free Entry

Let us go back to the model with linear demand and costs considered in Section 2. In this section we will assume that in order to produce, firms must incur in a fixed cost, denoted by \( K \), and that there is an infinite number of potential firms. The number of active firms in equilibrium is denoted by \( n^* \). Notice that for a given \( n^* \), output under free entry is identical to that calculated in the Introduction. Now we explain how \( n^* \) is determined. We will assume that the decision of entry is prior to the decision on output.\(^{11}\) Thus, equilibrium under free entry implies that if \( n^* \) firms are in the market,

\(^{10}\)The fact that WL are small in large economies under increasing returns has been shown by Dasgupta and Ushio (1981), Fraysse and Moreaux (1981) and Guesnerie and Hart (1985)

\(^{11}\)Lopez-Cuñat (1999) has shown that, under conditions that are met here, the equilibrium considered in this paper is a subset of equilibrium when both decisions are simultaneous (for the latter, see Novshek
firm \( n^* \) has non negative profits but firm \((n^* + 1)\) has non positive profits, formally
\[
\frac{a^2}{b(n^* + 1)^2} \leq K \leq \frac{a^2}{b(n^* + 2)^2}. \tag{5.1}
\]
The expression on the left (resp. right) hand side provides an upper (resp. lower) bound to the number of active firms, denoted by \( \tilde{n} \) (resp. \( n \)). Denoting \( Q \equiv a^2/bK \), we have \( \tilde{n} = \sqrt{Q} - 1 \) and \( n = \sqrt{Q} - 2 \). Clearly, the distance between \( \tilde{n} \) and \( n \) is 1, thus the number of active firms in equilibrium is unique so they are all other variables.\(^{12}\) Welfare in a Cournot equilibrium with free entry is
\[
W^* = \frac{a^2 n^* (n^* + 2)}{2b(n^* + 1)^2} - n^* K. \tag{5.2}
\]
where \( n^* \) is the unique number fulfilling (5.1). In an optimal allocation, aggregate output equals the one in (2.3) and, clearly, the number of active firms is one.\(^{13}\) Thus, social welfare is
\[
W^o = \frac{a^2}{2b} - K. \tag{5.3}
\]
The percentage of WL is
\[
PWL = 1 - n^* \frac{\frac{a^2 (n^* + 2)}{2b(n^* + 1)^2} - K}{\frac{a^2}{2b} - K} = 1 - n^* \frac{\frac{Q(n^* + 2)}{2(n^* + 1)^2} - 1}{\frac{Q}{2} - 1} \tag{5.4}
\]
It is easy to see that \( PWL \) is decreasing on \( Q \). Thus the maximum value of \( PWL \) occurs for the minimum value of \( Q \), namely, \( (n^* + 1)^2 \). Then (5.4) can be written as a function of \( n^* \) which is an observable variable. This gives us an upper bound to the WL, denoted by \( PWL \), which amounts to
\[
\overline{PWL} = 1 - \frac{n^*}{n^* + 2n^* - 1} = \frac{2n^* - 1}{n^* + 2n^* - 1}.
\]
Now we compute the minimum value of \( PWL \), denoted by \( \underline{PWL} \), which occurs for the maximum value of \( Q \), namely \( (n^* + 2)^2 \). Again, writing \( PWL \) in terms of \( n^* \), we have
\[
\underline{PWL} = \frac{2n^* + 3n^* + 2n^* + 2}{(n^* + 1)^2 (n^* + 4n^* + 2)}.
\]
\(^{12}\)If entry and output setting were simultaneous, Ushio (1983) has shown that there is a large number of equilibria when \( K \) is small.
\(^{13}\)Overentry may occur even if the marginal cost is increasing, see von Weizsäcker (1980).
The next table gives us maximum and minimum $PWL$ for selected values of $n^*$. 

<table>
<thead>
<tr>
<th>$n^*$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$PWL$</td>
<td>.43</td>
<td>.36</td>
<td>.30</td>
<td>.26</td>
<td>.23</td>
<td>.21</td>
<td>.19</td>
<td>.17</td>
<td>.16</td>
<td>.089</td>
</tr>
<tr>
<td>$PWL_{SB}$</td>
<td>.27</td>
<td>.24</td>
<td>.22</td>
<td>.20</td>
<td>.18</td>
<td>.17</td>
<td>.155</td>
<td>.144</td>
<td>.13</td>
<td>.08</td>
</tr>
</tbody>
</table>

**TABLE 2**

Notice the large values of WL for small values of $n$. But even for "large" values of $n^*$ WL are substantial: For instance for $n^* = 10$, the minimum $PWL$ is 13% which is larger than the WL in the case of no free entry with two firms.

As in the previous section, let us disentangle the effect from the misallocation of output from the effect of misallocation due to overentry. Computing the optimal allocation with the restriction that all equilibrium firms are active,

$$PWLSB = \frac{a^2 b}{2b(n^*+1)^2} - \frac{a^2 n^*(n^*+2)}{2b(n^*+1)^2}.$$ 

Clearly $PWLSB$ is decreasing on $K$. Recall that the maximum and the minimum values of $K$ are $\frac{a^2 b}{b(n^*+1)^2}$ and $\frac{a^2 b}{b(n^*+2)^2}$. Thus, the maximum and the minimum values of $PWLSB$, denoted by $PWLSB$ and $PWLSB_{SB}$ are, respectively,

$$PWLSB = \frac{1}{1+n^*} \quad \text{and} \quad PWLSB_{SB} = \frac{(n^*+2)^2}{(n^*+1)^2(n^*+3n^*+4)}.$$ 

Table 3 below summarizes the percentage of WL for selected values of $n^*$.

<table>
<thead>
<tr>
<th>$n^*$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$PWLSB$</td>
<td>.2</td>
<td>.1</td>
<td>.059</td>
<td>.038</td>
<td>.027</td>
<td>.015</td>
<td>.012</td>
<td>.01</td>
<td>.0025</td>
<td></td>
</tr>
<tr>
<td>$PWLSB_{SB}$</td>
<td>.127</td>
<td>.071</td>
<td>.045</td>
<td>.003</td>
<td>.0023</td>
<td>.0017</td>
<td>.0013</td>
<td>.001</td>
<td>.0009</td>
<td>.0002</td>
</tr>
</tbody>
</table>

**TABLE 3**
The percentage of maximum (resp. minimum) WL that can be attributed to oligopolistic output setting is the ratio between $PWL_{SB}$ and $PWL$ (resp. $PWL_{SB}$ and $PWL$).

Next table present these ratios for selected values of $n^*$. 

<table>
<thead>
<tr>
<th>$n^*$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$PWL_{SB}/PWL$</td>
<td>.46</td>
<td>.277</td>
<td>.20</td>
<td>.14</td>
<td>.117</td>
<td>.095</td>
<td>.079</td>
<td>.07</td>
<td>.0225</td>
<td>.003</td>
</tr>
<tr>
<td>$PWL_{SB}/PWL$</td>
<td>.477</td>
<td>.296</td>
<td>.205</td>
<td>.15</td>
<td>.127</td>
<td>.1</td>
<td>.083</td>
<td>.0694</td>
<td>.0692</td>
<td>.0025</td>
</tr>
</tbody>
</table>

TABLE 4

It is remarkable the similitude between the two columns. Only for the duopoly case both ratios are substantial. From five active firms on, the misallocation due to output is really negligible. 

The main conclusion of this section is that fixed costs and free entry imply a non-negligible misallocation of resources. And except if the number of firms is small, misallocation due to the wrong number of active firms are far greater than misallocation due to the wrong output being produced.

6. Non Identical Firms

Suppose that firms have different productivities. Let $c_i$ be the marginal cost of firm $i$. Without loss of generality let $c_1 \leq c_i$, for all $i$. Let $a_i \equiv A - c_i$. We will assume that for all $i$, $na_i > \sum_{j \neq i} a_j$. We will see that his assumption guarantees that all firms produce a positive output in equilibrium (see equation (6.1) below). Profits for firm $i$ can be written as $\pi_i = (a_i - bx)x_i$. It is easily calculated that in a Cournot equilibrium

$$x^*_i = \frac{a_i}{b} - \frac{\sum_{j=1}^{n} a_j}{b(n+1)} \quad \text{and} \quad x^* = \frac{\sum_{j=1}^{n} a_j}{b(n+1)}. \quad (6.1)$$

Social welfare is $W = Ax - \frac{bx^2}{2} - \sum_{j=1}^{n} c_jx_j = \sum_{j=1}^{n} a_jx_j - \frac{bx^2}{2}$. In a Cournot equilibrium

$$W^* = \sum_{i=1}^{n} a_i \left( \frac{a_i}{b} - \frac{\sum_{j=1}^{n} a_j}{b(n+1)} \right) - b \left[ \frac{\sum_{j=1}^{n} a_j}{b(n+1)} \right]^2 = \sum_{i=1}^{n} \frac{a_i^2}{b} - \frac{(2n+3)(\sum_{j=1}^{n} a_j)^2}{2b(n+1)^2}, \quad (6.2)$$
which when all $a_i$’s are identical reduces to (2.2). In the optimal allocation only the technology in the hands of Firm 1 is used and accordingly
\[ x^o = \frac{a_1}{b} \quad \text{and} \quad W^o = \frac{a_1^2}{2b}. \] (6.3)

Thus, the percentage of welfare loss can be written as
\[ PWL = \frac{a_1^2 - 2\sum_{i=1}^{n} a_i^2 + \frac{(2n+3)(\sum_{j=1}^{n} a_j)^2}{(n+1)^2}}{a_1^2}. \] (6.4)

In order to have a workable expression that depends on observable variables alone, let us define $s_i$ as the market share of firm $i$. Then,
\[ s_i = \frac{x_i}{x} = \frac{a_i(n+1) - \sum_{j=1}^{n} a_j}{\sum_{j=1}^{n} a_j} \Rightarrow a_i = \frac{(s_i + 1) \sum_{j=1}^{n} a_j}{n+1} \] (6.5)

Plugging the last part of (6.5) into (6.4) we obtain that
\[ PWL = \frac{\left(\frac{\sum_{j=1}^{n} a_j}{n+1}\right)^2 - 2\sum_{i=1}^{n} \left(\frac{\sum_{j=1}^{n} a_j}{n+1}\right)^2 + \frac{(2n+3)\left(\frac{\sum_{j=1}^{n} a_j}{n+1}\right)^2}{(n+1)^2}}{\left(\frac{\sum_{j=1}^{n} a_j}{n+1}\right)^2}. \] (6.6)

Dividing by $\left(\frac{\sum_{j=1}^{n} a_j}{n+1}\right)^2$ and rearranging, (6.6) simplifies to
\[ PWL = \frac{s_1^2 + 2s_1 - 2\sum_{i=1}^{n} (s_i + 1)^2 + 2n + 4}{(s_1 + 1)^2} = \frac{s_1^2 + 2s_1 - 2\sum_{i=1}^{n} s_i^2}{(s_1 + 1)^2} \] (6.7)

We see that in this case the percentage of welfare loss depends only on two things: on the market share of the dominant firm and on the Hirschman-Herfindahl index of concentration defined as $H = \sum_{i=1}^{n} s_i^2$. Notice that, for a given value of $H$, $PWL$ is increasing on $s_1$, which sounds reasonable: the larger the dominant firm, the closer to monopoly. However, for a given $s_1$, $PWL$ is decreasing on $H$! The reason is that when $H$ increases, production is being shifted to the less efficient firms which causes social welfare to fall.\footnote{In fact, social welfare is increasing in the marginal cost of small firms see Lahiri and Ono (1988). For a criticism of the idea that concentration is generally bad for social welfare see Daughety (1990) and Farrell and Shapiro (1990).}
We now study the extrema of (6.7). First we notice that since \( \sum_{i=1}^{n} s_i^2 = 1 \), (6.7) can be written as
\[
PWL = \frac{1 - (\sum_{i\neq 1} s_i)^2 - 2 \sum_{i\neq 1} s_i^2}{4 + (\sum_{i\neq 1} s_i)^2 - 4 \sum_{i\neq 1} s_i^2}.
\]
(6.8)

Now compute \( \frac{\partial PWL}{\partial s_j} \) for \( j = 2, \ldots, n \), which equals to
\[
-(4 + (\sum_{i\neq 1} s_i)^2 - 4 \sum_{i\neq 1} s_i)(2 \sum_{i\neq 1} s_i + 4s_j) + (2 \sum_{i\neq 1} s_i + 4)(1 - (\sum_{i\neq 1} s_i)^2 - 2 \sum_{i\neq 1} s_i^2)
\]
\[
(4 + (\sum_{i\neq 1} s_i)^2 - 4 \sum_{i\neq 1} s_i)^2
\]
From this equation we obtain the Kuhn Tucker inequalities corresponding to the extrema of \( PWL \) with respect to \( (s_2, s_3, \ldots, s_n) \) with inequality constraints \( \sum_{i\neq 1} s_i = 1 - s_1 \), and \( s_1 \geq s_j \geq 0 \), all \( j = 1, 2, 3, \ldots, n \). These are for \( j = 2, 3, \ldots, n \),
\[
\frac{\partial PWL}{\partial s_j} < 0 \rightarrow s_j = 0, \quad \frac{\partial PWL}{\partial s_j} > 0 \rightarrow s_j = 1, \quad \text{otherwise} \quad \frac{\partial PWL}{\partial s_j} = 0
\]
Comparing (6.9) corresponding to two different firms, say, \( i \) and \( k \), we see that the solution to (6.9) must be such that \( s_i = s_k = x \), say for all \( i \) and \( k \neq 1 \). For easiness of notation let \( m \equiv n - 1 \). Thus, FOC can be written as
\[
\frac{8mx^2 + 4m^2x^2 - 16x - 10mx + 4}{(4 + m^2x^2 - 4mx)^2}
\]
The inequality constraints read in this case \( s_1 = 1 - mx \) and \( s_1 \geq x \geq 0 \), which imply \( x(m + 1) \leq 1 \) or \( x \leq \frac{1}{m+1} \). Let us first find interior extrema of \( PWL \). This is equivalent to solving \( 4mx^2 + 2m^2x^2 - 8x - 5mx + 2 = 0 \). This equation has two solutions: \( x = \frac{m}{2} \) and \( x = \frac{1}{2m+4} \). The first solution is not acceptable since \( x \) must be not larger than \( \frac{1}{m+1} \). Thus extrema of \( PWL \) are, either \( x = 0 \), or \( x = \frac{1}{m+1} \) or \( x = \frac{1}{2m+4} \) which correspond to \( x = \frac{1}{n} \) or \( x = \frac{1}{2n+2} \). Plugging these values in (6.8) -which under symmetry reads
\[
\frac{1-m^2x^2-2mx^2}{4+m^2x^2-4mx} = \frac{x^2-n^2x^2+1}{4x-4nx+x^2-2nx^2+n^2x^2+4} = PWL(x),
\]
we obtain that
\[
PWL(0) = \frac{1}{4}, \quad PWL(\frac{1}{n}) = \frac{1}{(n+1)^2}, \quad PWL(\frac{1}{2n+2}) = \frac{n+1}{3n+5}.
\]
From these values of \( PWL \) we get the following:

1) The minimum \( PWL \) occurs when all firms are identical.
2) The maximum \( PWL \) occurs when \( s_1 = \frac{1}{2m+2} \) and \( s_2 = s_3 = \ldots = s_n = \frac{1}{2n+2} \)
(since \( 4(n + 1) > 3n + 5 \) for \( n > 1 \)).
7. Conclusions

When one observes public policies on oligopolies one sees some concern about the number of active firms. But the question of the output set by oligopolists is cause of little or no concern at all. This paper provides some justification to this attitude. In two different settings, namely decreasing marginal costs with a given number of firms and constant marginal costs plus a fixed cost and free entry, we found that welfare losses due to the wrong number of firms can be quite substantive even with as much as ten active firms in the market. On the contrary welfare losses due to the divergence between equilibrium and optimal output are small, even with as few as four firms in the market. This conclusion, though, is likely to be exaggerated by the fact that under our assumptions the optimal number of firms is one. It is possible that if an U-shaped average cost curve was assumed this conclusion would be weakened. Other factor likely to bring down our estimates of welfare losses is the consideration of other solution concepts (e.g. Bertrand). Thus our results are just a first cut to the problem.

Our results have a number of implications for the empirical literature that measures welfare losses. Typically, this literature has attempted to measure welfare losses arising from oligopolistic output setting. Our results suggest that this attempt is misguided because welfare losses arising from this source are likely to be small. However welfare losses due to the wrong number of firms can be quite substantial. At the best of my knowledge a measure of the latter has never been attempted.

References


