A point \( x \in A \subset (X, \| \cdot \|) \) is quasi-denting if for every \( \varepsilon > 0 \) there exists a slice of \( A \) containing \( x \) with Kuratowski index less than \( \varepsilon \). The aim of this paper is to generalize the following theorem with a geometric approach, see [19]: A Banach space such that every point of the unit sphere is quasi-denting (for the unit ball) admits an equivalent LUR norm.

1. Introduction

For a bounded set \( B \) in a metric space \( X \), the Kuratowski index of non-compactness of \( B \) is defined by

\[
\alpha(B) := \inf \{ \varepsilon > 0 : B \text{ is covered by a finite family of sets of diameter less than } \varepsilon \}
\]

Obviously \( \alpha(B) \leq \text{diam}(B) \) and \( \alpha(B) = 0 \) if, and only if, \( B \) is totally bounded in \( X \); i.e. relatively compact when \( X \) is a complete metric space.

If \( B \) is a closed convex and bounded subset of \( X \), a point \( x \in B \) is said to be quasi-denting for \( B \) if for every \( \varepsilon > 0 \) there exists an open half space \( H \) with \( x \in H \) and such that the ‘slice’ \( H \cap B \) has \( \alpha(H \cap B) < \varepsilon \). When in the former definition we require the diameter of \( H \cap B \) to be less than \( \varepsilon \) instead of \( \alpha(H \cap B) < \varepsilon \), the point \( x \) is said to be denting for \( B \). The notion of quasi-denting point was introduced in [4], under the name of \( \alpha \)-denting point, in connection with the investigation of differentiability properties of convex functions in Banach spaces; the notion of denting point goes back to the early studies of sets with the Radon-Nikodym property [1] and it was used in [18] to show that a Banach space \( X \) admits an equivalent locally uniformly rotund norm if all the points in its unit sphere \( S_X \) are denting points for the unit ball \( B_X \). For an elegant proof of this result see [17]. Let us recall that the norm \( \| \cdot \| \) in \( X \) is said to be locally uniformly rotund (LUR for short) if

\[
\lim_{n \to \infty} \| x_n - x \| = 0 \quad \text{whenever} \quad \lim_{n \to \infty} \left( 2\|x_n\|^2 + 2\|x\|^2 - \|x_n + x\|^2 \right) = 0.
\]

For an up-to-date account of LUR renormings we refer to [2, 5, 7, 20]. It is not known whether \( X \) admits an equivalent LUR norm if every bounded set in \( X \) has a slice of arbitrarily small diameter (i.e. the Radon-Nikodym property). G. Lancien proved that
$X$ admits an equivalent **LUR** norm whenever, for every $\varepsilon > 0$, $B_X$ is a union of complements of a decreasing transfinite but countable family $C^\varepsilon_\alpha$ of closed convex sets such that $C^\varepsilon_\alpha \setminus C^\varepsilon_{\alpha+1}$ is a union of slices of $C^\varepsilon_\alpha$ of diameter less than $\varepsilon$, [9, 10], see also [5].

Throughout this paper we shall denote by $X$ a normed space and $F \subset X^*$ will be a norming subspace for it; i.e. if we define

$$\|x\| := \sup\{|f(x)| : f \in B_{X^*} \cap F\}$$

for every $x \in X$,

then $\| \cdot \|$ provides an equivalent norm for $X$. When the original norm coincides with $\| \cdot \|$ we say that $F$ is 1-norming. As usual we denote by $\mathcal{H}(F)$ the family of all $\sigma(X,F)$-open half spaces in $X$. So for a point $x$ in a $\sigma(X,F)$-closed, convex and bounded subset $B$ of $X$, we shall say that $x$ is a $\sigma(X,F)$-denting ($\sigma(X,F)$-quasi-denting) for $B$ whenever the open half space in the definition of denting (quasi-denting) can be chosen from $\mathcal{H}(F)$.

The following modification of the ‘Cantor derivation’ is a main tool used by Lancien to obtain his result:

We fix a normed space $X$, a norming subspace $F \subset X^*$ and $B \subset X$ a $\sigma(X,F)$-closed, convex and bounded subset of $X$. Pick any $\varepsilon > 0$ and define

$$D_{\varepsilon,F} (B) := \{x \in B : \| \cdot - \text{diam} (H \cap B) > \varepsilon \text{ for all } H \in \mathcal{H}(F), x \in H\}$$

Again $D_{\varepsilon,F} (B)$ is a $\sigma(X,F)$-closed, convex and bounded subset of $X$ and we can iterate the construction and define

$$D_{\varepsilon,F}^{\alpha+1} (B) := D_{\varepsilon,F} (D_{\varepsilon,F}^\alpha (B)),$$

where $D_{\varepsilon,F}^0 (B) := B$

and

$$D_{\varepsilon,F}^\alpha (B) := \bigcap_{\beta < \alpha} D_{\varepsilon,F}^\beta (B) \text{ if } \alpha \text{ is a limit ordinal.}$$

Then we set

$$\delta_F (B, \varepsilon) := \begin{cases} \inf \{\alpha : D_{\varepsilon,F}^\alpha (B) = \emptyset\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

and $\delta_F (B) := \sup\{\delta_F (B, \varepsilon) : \varepsilon > 0\}$.

Indeed, Lancien showed that $\delta_X (B_X) < \omega_1$ (resp. $\delta_X (B_{X^*}) < \omega_1$) implies $X$ (resp. $X^*$) admits an equivalent **LUR** norm (resp. dual **LUR** norm). For quasi-denting points, refining probabilistic methods, it was shown in [19] that a Banach space $X$ also admits an equivalent **LUR** norm provided all points in the unit sphere $S_X$ are quasi-denting points for $B_X$. A modification of the derivation approach has been subsequently used by M. Raja [17] who provided a transparent proof, and significative improvements, of the result of the fourth author for denting points, [18]. Nevertheless, for quasi-denting points the approach is still fully probabilistic. A first contribution in this paper will be to provide Raja’s approach for quasi-denting points; i.e. to show with a geometric construction, free from probabilistic arguments, the theorem for quasi-denting points. Indeed, for a given subset $S$ of a $\sigma(X,F)$-closed, convex and bounded subset $B$ of a
normed space $X$ we define its dentability index (with respect to the norming subspace $F \subset X^*$) in $B$ as follows:

$$\delta_F(S, B, \varepsilon) := \begin{cases} \inf \{\alpha : D_{\varepsilon, F}(B) \cap S = \emptyset\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

and $\delta_F(S, B) := \sup \{\delta_F(S, B, \varepsilon) : \varepsilon > 0\}$.

In other words we want to measure how many steps of Lancien’s derivation process for $B$ are necessary to “eat out” the subset $S$. When all the points of the unit sphere are denting points for the unit ball of $X$ we obviously have $\delta_X(S, B_X) = 1$ and Raja’s approach immediately gives the following:

**Theorem 1.** ([17]) If $\delta_F(S_X, B_X) < \omega_1$ the normed space $X$ admits an equivalent $\sigma(X, F)$-lower semi-continuous LUR norm.

Indeed, the following is a tool for LUR renormings, [15].

**Theorem 2.** ([12, 17]) Let $X$ be a normed space and let $F$ be a norming subspace of its dual. Then $X$ admits an equivalent $\sigma(X, F)$-lower semi-continuous LUR norm if, and only if, for every $\varepsilon > 0$ we can write

$$X = \bigcup_{n=1}^{\infty} X_{n, \varepsilon}$$

in such a way that for every $x \in X_{n, \varepsilon}$, there exists a $\sigma(X, F)$-open half space $H$ containing $x$ with diam $(H \cap X_{n, \varepsilon}) < \varepsilon$.

When in the above theorem we replace open slices with weak ($\sigma(X, F)$) open sets we arrive to the concept introduced in [8]. Namely: a normed space $X$ is said to have a countable cover by sets of small local diameter if for every $\varepsilon > 0$,  

$$X = \bigcup_{n=1}^{\infty} X_{n, \varepsilon}$$

in such a way that for every $x \in X_{n, \varepsilon}$ there exists a weak ($\sigma(X, F)$) open set $W$ containing $x$ with diam $(W \cap X_{n, \varepsilon}) < \varepsilon$. One could replace the diameter in the definition above by Kuratowski index of non-compactness to measure the size of the set $W \cap X_{n, \varepsilon}$. In this case, since closed balls are weak ($\sigma(X, F)$)-closed, one can easily show that both notions coincide.

Let us recall that a normed space has the Kadec property if the norm and weak topologies coincide on the unit sphere. Using that an extreme point of continuity is denting ([11]) we reformulate the theorem mentioned before: a rotund Banach space with the Kadec property is LUR renormable. It is well known that $\ell_\infty$ has a rotund norm but fails to have a norm with the Kadec property. R. Haydon [7] proved that $C(\mathcal{T})$, $\mathcal{T}$ diadic tree, admits a norm with the Kadec property but fails to have a rotund norm if the height of the tree is bigger or equal to $\omega_1$. However, both Kadec property and rotundity in different combinations can be replaced by something weaker. In [12] is shown that if $X$ has a countable cover by sets of small local diameter and all points of $S_X$ are extreme for
then $X$ admits a LUR norm. In [14] is shown that if $X$ has the Kadec property and all faces of its unit sphere have Krein-Milman property then it admits a LUR renorming.

Our main results in this paper will provide extensions of the former results when we use the Kuratowski index of non-compactness instead of the diameter both in the derivation process of Lancien and in the statement of the former theorem. Indeed, we shall prove the following:

**Theorem 3.** Let $X$ be a normed space and let $F$ be a norming subspace of its dual. Then $X$ admits an equivalent $\sigma(X, F)$-lower semi-continuous LUR norm if, and only if, for every $\varepsilon > 0$ we can write

$$X = \bigcup_{n=1}^{\infty} X_{n, \varepsilon}$$

in such a way that for every $x \in X_{n, \varepsilon}$, there exists a $\sigma(X, F)$-open half space $H$ containing $x$ with $\alpha (H \cap X_{n, \varepsilon}) < \varepsilon$.

Taking advantage of homogeneity one can replace the space $X$ in the former theorem by its unit sphere, see theorem 11.

In the course of the proof we shall show that a $\sigma(X, F)$-closed, convex and bounded subset $B$ of the normed space $X$ has $\sigma(X, F)$-open slices of arbitrarily small diameter if, and only if, it has $\sigma(X, F)$-open slices of arbitrarily small Kuratowski index (corollary 5), and therefore the index of non-compactness also gives characterizations of sets with the Radon-Nikodým property; moreover, we shall show that

$$\delta_F (\sigma(X, F) - \text{quasi-denting points of } B, B) < \omega_1$$

from where the theorem of the fourth named author [19] follows immediately (corollary 10).

From the topological point of view we shall see that a normed space $X$ admits an equivalent $\sigma(X, F)$-lower semi-continuous LUR norm if, and only if, the norm topology has a network $\mathcal{N} = \cup_{n=1}^{\infty} \mathcal{N}_n$ such that for every $n \in \mathbb{N}$ and for every $x \in \cup \mathcal{N}_n$ there is a $\sigma(X, F)$-open half space $H$ with $x \in H$ such that $H$ meets only a finite number of elements from $\mathcal{N}_n$ (corollary 13), therefore turning the ‘discrete’ condition appearing in [13] into a ‘locally finite’ one.

Throughout the paper when dealing with a normed space $X$ and $F \subset X^*$ a norming subspace for it, in order to simplify the notation, all closures taken in $X$ will be with respect to the $\sigma(X, F)$-topology unless otherwise stated.

2. **Kuratowski’s Index of Non-Compactness and Dentability**

In order to work with the index of non-compactness we need to introduce the following definition for a bounded subset $B$ of a normed space $X$ and a positive integer $p$:

$$\alpha (B, p) := \inf \{ \varepsilon > 0 : B \text{ is covered by } p \text{ sets of diameter less than } \varepsilon \}$$

and we have $\alpha (B) = \inf \{ \alpha (B, p) : p = 1, 2, \ldots \}$.

The first result we need is the following:
Lemma 1. Let $X$ be a normed space and $F \subset X^*$ be a 1-norming subspace. Let $C$, $C_0$ and $C_1$ be $\sigma(X, F)$-closed, convex and bounded subsets of $X$. Let $p$ be a positive integer, $\varepsilon > 0$ and $M = \text{diam}(C_0 \cup C_1)$. If we assume that

1. $C_0 \subset C$ and $\alpha(C_0, p) < \varepsilon'$;
2. $C$ is not a subset of $C_1$;
3. $C$ is a subset of $\overline{C^*}(C_0, C_1)$.

Then if $r$ is a positive number such that $2rM + \varepsilon' < \varepsilon$ and we set

$$D_r := \{(1 - \lambda)x_0 + \lambda x_1 : r \leq \lambda \leq 1, x_0 \in C_0, x_1 \in C_1\},$$

then

$$C \setminus D_r \neq \emptyset \text{ and } \alpha(C \setminus D_r, p) < \varepsilon.$$

Remark 1. For $p = 1$, the lemma above is just the Bourgain-Namioka superlemma, see [1, 3]. Following the proof of that case it is not difficult to see that it remains true for every $p \in \mathbb{N}$. Since we shall make use of some of details from the proof we shall give it in full detail.

Proof. For $0 \leq r \leq 1$ we define

$$D_r := \{(1 - \lambda)x_0 + \lambda x_1 : r \leq \lambda \leq 1, x_0 \in C_0, x_1 \in C_1\}.$$

Note that $D_r$ is convex, $D_0 \supseteq C$ (by condition 3), $D_1 = C_1$ and for $0 < r < 1$ we have $D_r \not\subseteq C$. Let us show the last claim.

Since $C \not\subseteq C_1$ we can find $x^* \in F$ such that $\sup x^*(C_1) < \sup x^*(C)$. Now if $C$ were contained in $D_r$ for some $r > 0$, then we would have

$$\sup x^*(C) \leq \sup x^*(D_r) = \sup x^*(D_r) \leq$$

$$\leq (1 - r) \sup x^*(C_0) + r \sup x^*(C_1) \leq (1 - r) \sup x^*(C) + r \sup x^*(C_1)$$

thus $r \sup x^*(C) \leq r \sup x^*(C_1)$, and since $r > 0$ we would have $\sup x^*(C) \leq \sup x^*(C_1)$, a contradiction.

Since $\alpha(C_0, p) < \varepsilon'$ we should have $C_0 \subset \bigcup_{i=1}^p B_i$ with $\text{diam}(B_i) < \varepsilon'$. Finally fix $r > 0$ such that $2rM + \varepsilon' < \varepsilon$.

Notice that $C \setminus D_r \subset D_0 \setminus D_r$. Also

$$D_0 \setminus D_r \subset C_0 + B(0; rM) \subset \bigcup_{i=1}^p [B_i + B(0; rM)].$$

Indeed, if $x \in D_0 \setminus D_r$, $x = (1 - \lambda)x_0 + \lambda x_1$ with $x_0 \in C_0, x_1 \in C_1$ and $0 \leq \lambda < r$. Then

$$\|x - x_0\| = \lambda \|x_0 - x_1\| < rM.$$

Now

$$D_0 \setminus D_r \subset \bigcup_{i=1}^p B_i + B(0; rM)$$

and the sets $B_i + B(0; rM)$ have diameter less than $2rM + \varepsilon' < \varepsilon$. \hfill \Box

An easy consequence of lemma 1 for $p = 1$ is the following.
Proposition 4. Let $X$ be a normed space and $F \subset X^*$ be 1-norming subspace. If $B$ is a $\sigma(X,F)$-closed, convex and bounded subset of $X$ and $H$ is a $\sigma(X,F)$-open half space with $H \cap B \neq \emptyset$ and $\alpha(H \cap B) < \varepsilon$, then there exists another $\sigma(X,F)$-open half space $G$ with $\emptyset \neq G \cap B \subset H \cap B$ and $\text{diam}(G \cap B) < \varepsilon$.

Proof. By induction on the integer $p$ such that $\alpha(H \cap B, p) < \varepsilon$. For $p = 1$ there is nothing to prove. Let us assume the assertion is true for $p \leq n - 1$ and write

$$H \cap B \subset B_1^H \cup B_2^H \cup \ldots \cup B_n^H$$

where every $B_i^H$ is a $\sigma(X,F)$-closed and convex set with $\text{diam}(B_i^H) < \varepsilon$. If we define

$$L_1 := \overline{\text{co}}(B \setminus H, B_1^H \cap B)$$

we have two possibilities:

(i) $B = L_1$ and we can apply lemma 1 for $p = 1$ to the sets $C_0 = B_1^H \cap B$ and $C_1 = B \setminus H$ to obtain a $\sigma(X,F)$-open half space $G$ with $G \cap B_1^H \neq \emptyset, G \cap B \subset H \cap B$ and $\text{diam}(G \cap B) < \varepsilon$.

(ii) $L_1 \subsetneq B$, then for any $y \in B \setminus L_1$ we have, by Hahn-Banach’s Theorem, a $\sigma(X,F)$-open half space $\tilde{H}$ with $y \in \tilde{H}$ and $\tilde{H} \cap B \subset H \cap B$ but

$$\tilde{H} \cap B \subset B_2^H \cup B_3^H \cup \ldots \cup B_n^H$$

and if we apply the induction hypothesis to this slice the proof is done. □

As a corollary we obtain a better statement than the one given by Gilles and Moors in [4], see Theorems 4.2 and 4.3.

Corollary 5. For a normed space $X$, a norming subspace $F \subset X^*$ and a $\sigma(X,F)$-closed, convex and bounded subset $B$ of $X$, the following are equivalent:

1. $B$ is $\sigma(X,F)$-dentable; i.e. $B$ has $\sigma(X,F)$-open slices of arbitrarily small diameter;
2. $B$ has $\sigma(X,F)$-open slices whose Kuratowski index of non-compactness is arbitrarily small.

Proof. We can work with the equivalent norm $|| \cdot ||$ given by the norming subspace $F$ and apply proposition 4 for every $\varepsilon > 0$. □

3. Dentability index of quasi-denting points

We are going to iterate now Bourgain-Namioka superlemma together with the former construction in proposition 4 to describe when quasi-denting points are eaten out in Lancien’s derivation process. For a normed space $X$ and a norming subset $F \subset X^*$, we shall denote by $\mathbb{H}(F)$ the family of all $\sigma(X,F)$-open half spaces in $X$. Indeed, we shall prove the following:

Theorem 6. For every $\varepsilon > 0$ there is a countable ordinal $\eta_\varepsilon$ such that if $X$ is a normed space and $F \subset X^*$ is a 1-norming subspace, then for every $\sigma(X,F)$-closed, convex and bounded subset $B$ of $X$, if

$$Q_\varepsilon := \{ x \in B; \exists H \in \mathbb{H}(F), x \in H \text{ with } \alpha(H \cap B) < \varepsilon \},$$
then we have
\[ \delta_F(Q, B, \varepsilon) < \eta < w \]

The proof of the theorem is based on a series of previous results. We begin with:

**Lemma 2.** Let \( B \) be \( \sigma(X, F) \)-closed, convex and bounded subset of \( X \), where \( F \) is a 1-norming subspace for \( X \) and \( \varepsilon > 0 \) be fixed. Let \( B := L_0 \supset L_1 \supset L_2 \supset \ldots \supset L_n \) be \( \sigma(X, F) \)-closed and convex. Let \( S \) be a subset of \( L_n \), then
\[
\delta_F(S, B, \varepsilon) \leq \delta_F(L_0 \setminus L_1, B, \varepsilon) + \delta_F(L_1 \setminus L_2, L_1, \varepsilon) + \ldots
\]
\[ + \delta_F(L_{n-1} \setminus L_n, L_{n-1}, \varepsilon) + \delta_F(S, L_n, \varepsilon) \]

**Proof.** We shall prove it by induction on \( n \). For \( n = 1 \) let \( B = L_0 \supset L_1 \supset S \) be as in the statement and let: \( \delta_F(L_0 \setminus L_1, B, \varepsilon) = \alpha, \delta_F(S, L_1, \varepsilon) = \beta \).

Since \( D_{\varepsilon,F}^n(B) \cap (B \setminus L_1) = \emptyset \) we have \( D_{\varepsilon,F}^n(B) \subset L_1 \). Given \( x \in D_{\varepsilon,F}^{n+1}(B) \), we have \( \text{diam}(H \cap D_{\varepsilon,F}^n(B)) > \varepsilon \) for every \( H \in \mathbb{H}(F) \), so \( \text{diam}(H \cap L_1) > \varepsilon \) for every \( H \in \mathbb{H}(F) \), thus \( x \in D_{\varepsilon,F}(L_1) \). So
\[
D_{\varepsilon,F}^{n+1}(B) \cap S \subset D_{\varepsilon,F}(L_1) \cap S.
\]
Since \( \beta \) is the first ordinal such that \( D_{\varepsilon,F}^n(B) \cap S = \emptyset \) one must have \( D_{\varepsilon,F}^{n+\beta}(B) \cap S = \emptyset \), therefore \( \delta_F(S, B, \varepsilon) \leq \alpha + \beta \).

Now suppose we have \( B := L_0 \supset L_1 \supset L_2 \supset \ldots \supset L_n \supset S \) as in the statement and suppose the formula holds for \( n - 1 \) sets. Considering \( L_0 \supset L_1 \supset S \), as we did before,
\[
\delta_F(S, B, \varepsilon) \leq \delta_F(L_0 \setminus L_1, B, \varepsilon) + \delta_F(S, L_1, \varepsilon) \quad (*)
\]
If we consider now the sets \( L_1 \supset L_2 \supset \ldots \supset L_n \supset S \), by the induction hypothesis
\[
\delta_F(S, L_1, \varepsilon) \leq \delta_F(L_1 \setminus L_2, L_1, \varepsilon) + \ldots + \delta_F(L_{n-1} \setminus L_n, L_{n-1}, \varepsilon) + \delta_F(S, L_n, \varepsilon)
\]
To finish the proof we just need to use the later inequality in \((*)\). \( \square \)

**Lemma 3.** Let \( B \) be \( \sigma(X, F) \)-closed, convex and bounded subset of a normed space \( X \), where \( F \) is a 1-norming subspace for \( X \) and \( \varepsilon > 0 \) be fixed. Let \( H \) be a \( \sigma(X, F) \)-open half space with
\[
\alpha(H \cap B, n) < \varepsilon \text{ for some } n > 1 \text{ fixed.}
\]
Then there exists a sequence of \( \sigma(X, F) \)-closed, convex subsets
\[
B =: B_0 \supset B_1 \supset B_2 \supset \ldots \supset B_s \supset B_{s+1} \supset \ldots
\]
such that
\[
H \cap B \subset (B_0 \setminus B_1) \cup (B_1 \setminus B_2) \cup \ldots \cup (B_s \setminus B_{s+1}) \cup \ldots
\]
and for every \( s = 0, 1, 2, \ldots \) and every \( y \in B_s \setminus B_{s+1} \) there is a \( \sigma(X, F) \)-open half space \( G \) with \( y \in G, G \cap B \subset H \cap B, \) and
\[
\alpha(G \cap B_s, p) < \varepsilon \text{ for some } p \leq n - 1
\]
Proof. Since $\alpha(H \cap B, n) < \varepsilon$ we can fix $\sigma(X, F)$-closed, convex non-void sets

$$\{B_1^H, B_2^H, \ldots, B_n^H\}$$

with $\text{diam}(B_i^H) < \varepsilon$, for $i = 1, 2, \ldots, n$

and $H \cap B \subset B_1^H \cup \ldots \cup B_n^H$.

Let us define

$$L_1 := \text{cl}(B \setminus H, B_1^H \cap B).$$

If $y \in B \setminus L_1$, Hahn-Banach’s Theorem provides us with a $\sigma(X, F)$-open half space $G$, with $y \in G$ and $G \cap L_1 = \emptyset$, thus

$$G \cap B \subset H \cap B \text{ and } G \cap B \subset B_2^H \cup \ldots \cup B_n^H$$

and therefore $\alpha(G \cap B, p) < \varepsilon$ for some $p \leq n - 1$.

Let us consider the sets $C_0^1 := B_1^H \cap B$ and $C_1 = B \setminus H$ and apply lemma 1 with $p = 1$, to find $0 < r < 1$, indeed it is enough if $2r \text{diam}(B) + \text{diam}(B_1^H) < \varepsilon$, such that if

$$D_{r,1} := \{(1 - \lambda)x_0 + \lambda x_1; r \leq \lambda \leq 1, x_0 \in C_0^1, x_1 \in C_1\}$$

we have $L_1 \setminus D_{r,1} \neq \emptyset$ and $\text{diam}(L_1 \setminus D_{r,1}) < \varepsilon$. So for every $y \in L_1 \setminus D_{r,1}$ we should have a $\sigma(X, F)$-open half space $G$ with $y \in G$, $G \cap D_{r,1} = \emptyset$, thus $G \cap B \subset H \cap B$ and $G \cap L_1 \subset L_1 \setminus D_{r,1}$, so $\text{diam}(G \cap L_1) < \varepsilon$ and $\alpha(G \cap L_1, 1) < \varepsilon$.

We set $B_1 := L_1$ and $B_2 := D_{r,1}$. We shall iterate now the former construction to “eat out” the whole $B_1^H$ and to reach all the points of $B \cap H$ in a countable number of steps.

Let us define

$$L_2 := \text{cl}(B \setminus H, B_1^H \cap D_{r,1})$$

If $y \in D_{r,1} \setminus L_2$, there is a $\sigma(X, F)$-open half space $G$ with $y \in G$ and $G \cap L_2 = \emptyset$, thus $G \cap B \subset H \cap B \subset B_2^H \cup \ldots \cup B_n^H$. Moreover

$$G \cap D_{r,1} \subset B_1^H \cup \ldots \cup B_n^H$$

since $D_{r,1} \cap B_1^H \subset L_2$, and then $\alpha(G \cap D_{r,1}, p) < \varepsilon$ for some $p \leq n - 1$.

We shall now apply again the Bourgain-Namioka superlemma with the sets

$$C_0^2 := B_1^H \cap D_{r,1} \text{ and } C_1 := B \setminus H$$

and with the same $r$ as above we should have $\text{diam}(L_2 \setminus D_{r,2}) < \varepsilon$ where

$$D_{r,2} := \{(1 - \lambda)x_0 + \lambda x_1; r \leq \lambda \leq 1, x_0 \in C_0^2, x_1 \in C_1\}.$$ 

As before, for every $y \in L_2 \setminus D_{r,2}$ there exists a $\sigma(X, F)$-open half space $G$ with $y \in G$, $G \cap B \subset H \cap B$ and $\alpha(G \cap L_2, 1) < \varepsilon$.

We set $B_3 := L_2$ and $B_4 := D_{r,2}$. The process will continue by induction defining a sequence of sets

$$B = B_0 \supset L_1 \supset D_{r,1} \supset L_2 \supset D_{r,2} \supset \ldots \supset L_s \supset D_{r,s} \supset L_{s+1} \supset \ldots$$

such that for every $y \in L_s \setminus D_{r,s}$ there is a $\sigma(X, F)$-open half space $G$ with $y \in G$, $G \cap B \subset H \cap B$ and $\alpha(G \cap L_s, 1) < \varepsilon$; and for every $y \in D_{r,s} \setminus L_{s+1}$ there is a $\sigma(X, F)$-open half space $G$ with $y \in G, G \cap B \subset H \cap B$ and $\alpha(G \cap D_{r,s}, p) < \varepsilon$ for some $p \leq n - 1$. 

Proposition 7. Whenever in that case too. For the first step
$$\sup f|_{\mathcal{B},r} \leq (1 - r) \sup f(B_1^H \cap B) + r \mu$$
for the second step and by recurrence
$$\sup f|_{\mathcal{B},r} \leq (1 - r)^s \sup f(B_1^H \cap B) + r \mu [1 + (1 - r) + \ldots + (1 - r)^{s-1}]$$
for $s = 1, 2, \ldots$. Consequently for every $y$ with $f(y) > \mu$, $y$ cannot be in all the sets $\mathcal{B},r$ for $s = 1, 2, \ldots$ because the former inequality should imply $f(y) \leq \mu$. Then if $s$ is the first integer with $y \notin \mathcal{B},r$, we will have either $y \in L_s \setminus \mathcal{B},r$ or $y \in \mathcal{B},r,s-1 \setminus L_s$, when $s \geq 2$ and $y \in B \setminus L_1$ or $y \in L_1 \setminus \mathcal{B},r$ when $s = 1$.

The lemma is finished by defining $B_{2n+1} := L_{n+1}$ and $B_{2n} := \mathcal{B},r,n, n = 1, 2, \ldots$ when the process does not stop and $B_{s_0} = \mathcal{B},r,s_0, B_{s_0+1} = \ldots = \emptyset$ when the process stops at the $s_0$-step. We have seen before that $\alpha (H \cap \mathcal{B},r,s_0, p) < \varepsilon$ for some $p \leq n - 1$ in that case too.

$\square$

**Proposition 7.** For every $\varepsilon > 0$, there exists a sequence of ordinal numbers

$$1 =: \xi_1 < \xi_2 < \ldots < \xi_p < \ldots < \omega_1$$

such that if $X$ is a normed space and $F \subset X^*$ is a 1-norming subspace for it, we have

$$\delta_F (H \cap B, B, \varepsilon) \leq \xi_p \quad (\ast)$$

whenever $B$ is a $\sigma(X, F)$-closed, convex and bounded subset of $X$ and $H$ is a $\sigma(X, F)$-open half space with $\alpha (H \cap B, p) < \varepsilon$.

**Proof.** We shall define by induction on $p$ the sequence of countable ordinals $(\xi_n)_n$. For $p = 1$ the Kuratowski index $\alpha(\cdot, 1)$ coincides with the diameter and $H \cap B$ should be *eaten* at the first step of the derivation process, i.e., $\xi_1 := 1$ verifies $(\ast)$.

Let us assume that we have already defined

$$\xi_1 < \xi_2 < \ldots < \xi_{n-1} < \omega_1$$

such that $(\ast)$ is satisfied for $p \leq n - 1$. Let us fix a $\sigma(X, F)$-closed, convex and bounded subset of $X$ and $H$ a $\sigma(X, F)$-open half space with

$$\alpha (H \cap B, n) < \varepsilon$$
By lemma 3 we have a sequence of \( \sigma(X,F) \)-closed, convex subsets

\[
B = B_0 \supset B_1 \supset \ldots \supset B_s \supset B_{s+1} \supset \ldots \quad (**)
\]

such that

\[
H \cap B \subset (B_0 \setminus B_1) \cup (B_1 \setminus B_2) \cup \ldots \cup (B_s \setminus B_{s+1}) \cup \ldots
\]

and for every \( s \) and every \( y \in B_s \setminus B_{s+1} \) there exists a \( \sigma(X,F) \)-open half space \( G \), with \( y \in G \), \( G \cap B \subset H \cap B \), and \( \alpha (G \cap B, p) < \varepsilon \) for some \( p \leq n - 1 \). By our induction assumption we should have

\[
\delta_F (G \cap B_s, B_s, \varepsilon) \leq \xi_{n-1}
\]

and consequently \( \delta_F (B_s \setminus B_{s+1}, B_s, \varepsilon) \leq \xi_{n-1} \), \( s = 0, 1, 2, \ldots \) when \( (**) \) is infinite and \( \delta_F (H \cap B_{s_0}, B_{s_0}, \varepsilon) \leq \xi_{n-1} \) too, when the sequence stops at the \( s_0 \)-step. In any case we can apply lemma 2 to obtain

\[
\delta_F (B_s \setminus B_{s+1}, B, \varepsilon) \leq (s + 1)\xi_{n-1} \quad \text{for} \quad s = 0, 1, 2, \ldots
\]

Therefore we have

\[
\delta_F (H \cap B, B, \varepsilon) \leq \sup \{ (s + 1)\xi_{n-1} : s = 0, 1, 2, \ldots \} =: \xi_n
\]

which finishes the induction process. 

\[\square\]

**Corollary 8.** For every \( \sigma(X,F) \)-closed, convex and bounded subset \( B \) of \( X \), if

\[
Q_{\varepsilon,p} := \{ x \in B : \exists H \in \mathbb{H}(F), x \in H \text{ with } \alpha (H \cap B, p) < \varepsilon \}
\]

then we have

\[
\delta_F (Q_{\varepsilon,p}, B, \varepsilon) \leq \xi_p < \omega_1, \quad p = 1, 2, \ldots
\]

**Proof.** \( Q_{\varepsilon,p} = \cup \{ H \cap B : H \in \mathbb{H}(F) \text{ and } \alpha (H \cap B, p) < \varepsilon \} \) and

\[
\delta_F (Q_{\varepsilon,p}, B, \varepsilon) \leq \sup \{ \delta_F (H \cap B) : H \in \mathbb{H}(F) \text{ and } \alpha (H \cap B, p) < \varepsilon \} \leq \xi_p
\]

\[\square\]

Now we arrive to the **proof of theorem 6.**

We have \( Q_\varepsilon = \cup \{ Q_{\varepsilon,p} : p = 1, 2, \ldots \} \), by the former corollary we see that

\[
\delta_F (Q_{\varepsilon,p}, B, \varepsilon) \leq \xi_p \quad \text{for} \quad p = 1, 2, \ldots \quad \text{from where it follows}
\]

\[
\delta_F (Q_\varepsilon, B, \varepsilon) \leq \sup \{ \xi_p : p = 1, 2, \ldots \} =: \eta_\varepsilon < \omega_1
\]

because \( \omega_1 \) is not the limit of a sequence of countable ordinals.

\[\square\]

**Corollary 9.** There is a countable ordinal \( \eta \) such that if \( X \) is a normed space and \( F \subset X^* \) is a norming subspace, then for every \( B \subset X \) a \( \sigma(X,F) \)-closed convex and bounded subset of \( X \), if \( Q \) is the sets of quasi-denting points of \( B \), we have

\[
\delta_F (Q, B) < \eta < \omega_1
\]
Proof. It is not a restriction to assume that the given norm is \( \| \cdot \| \), making \( F \) a \( 1 \)-norming subspace, then we have

\[
\delta_F (Q, B) \leq \sup \{ \eta_{\varepsilon_n} ; n = 1, 2, \ldots \} =: \eta < \omega_1
\]

where \( \varepsilon_n \) tends to 0. \( \square \)

From theorem 1 in the introduction we get the theorem of the fourth named author with a geometric proof in full generality:

**Corollary 10.** If the normed space \( X \) has a norming subspace \( F \subset X^* \) such that \( S_X \) is formed by quasi-denting points of \( B_X \), in \( \sigma(X, F) \), then \( \delta_F (S_X, B_X) < \omega_1 \) and consequently \( X \) admits an equivalent \( \sigma(X, F) \)-lower semi-continuous **LUR** norm.

4. **LUR Renorming Theorem**

The aim of this section is to prove the following result, from where theorem 3 in the introduction is a particular case. Let us recall that a subset \( A \subset X \) of the normed space \( X \) is said to be a **radial set** if for every \( x \in X \) there is \( \rho > 0 \) such that \( \rho x \in A \).

**Theorem 11.** Let \( X \) be a normed space and \( F \subset X^* \) be a norming subspace for it. The following conditions are equivalent:

1. \( X \) admits an equivalent \( \sigma(X, F) \)-lower semi-continuous **LUR** norm;
2. For every \( \varepsilon > 0 \), \( X = \bigcup \{ X_{n, \varepsilon} : n \in \mathbb{N} \} \) such that for every \( n \in \mathbb{N} \) and \( x \in X_{n, \varepsilon} \) there exists \( H \), \( \sigma(X, F) \)-open half space with \( x \in H \) and \( \alpha(H \cap X_{n, \varepsilon}) < \varepsilon \);
3. There exists a radial set \( A \subset X \) such that for every \( \varepsilon > 0 \), \( A = \bigcup \{ A_{n, \varepsilon} : n \in \mathbb{N} \} \) such that for every \( n \in \mathbb{N} \) and \( x \in A_{n, \varepsilon} \) there exists \( H \), \( \sigma(X, F) \)-open half space with \( x \in H \) and \( \alpha(H \cap A_{n, \varepsilon}) < \varepsilon \).

Let us observe that no convex assumption is required for the sets \( \{ X_{n, \varepsilon} \} \) or \( \{ A_{n, \varepsilon} \} \) in the decompositions above. As for the proof of theorem 2 in the introduction, see [17], we need firstly a convexification argument that will reduce theorem 11 to theorem 2 because of the study we have done in the previous section.

We begin with a revision of lemma 1 for an arbitrary set \( A \) and \( x \in A \) with a half space \( H \in \mathbb{H}(F) \) with \( x \in H \) and \( \alpha(H \cap A) < \varepsilon \) (in this case we shall say that \( x \) is an \( \varepsilon \)-\( \sigma(X, F) \)-quasi-denting point for \( A \)).

**Lemma 4.** Let \( A \) be a bounded subset of the normed space \( X \), \( F \subset X^* \) be \( 1 \)-noring for it. Set \( M := \text{diam} (A) \) and let \( \varepsilon > 0 \) be fixed. If \( x \in A \) is such that there is \( H = \{ y \in X : g(y) > \eta \} \), with \( g \in F \), \( \eta \in \mathbb{R} \), \( x \in H \) and \( \alpha(H \cap A) < \varepsilon \), then there exists \( r \in [0, 1] \) which only depends upon \( \varepsilon \) and \( M \) such that we can fix a \( \sigma(X, F) \)-closed and convex subset \( D_{r}(x) \subset \overline{\sigma}(A) \) with the following properties:

1. \( \overline{\sigma}(A) \setminus D_{r}(x) \neq \emptyset \);
2. \( \alpha(\overline{\sigma}(A) \setminus D_{r}(x)) < 3\varepsilon \);
3. \( \sup g(D_{r}(x)) \leq (1 - r) \sup g(\overline{\sigma}(A)) + r\eta \).
Proof. Let us choose sets $B_1, B_2, \ldots, B_n$ with diam $(B_i) < \varepsilon$ and $u_i \in B_i \cap A$, $i = 1, \ldots, n$, such that $H \cap A \subset \cup \{B_i : i = 1, \ldots, n\}$. Now let $K_\varepsilon := \text{co} (u_1, \ldots, u_n)$ and set

$$C_0 := \{ y \in \overline{\text{aff}}(A) : \text{dist} (y, K_\varepsilon) \leq \varepsilon \}$$

and

$$C_1 := \{ y \in \overline{\text{aff}}(A) : g(y) \leq \eta \} = \overline{\text{aff}}(A) \setminus H.$$ 

As we did in lemma 1, for $0 \leq r \leq 1$, let

$$D_r := \{(1 - \lambda)x_0 + \lambda x_1 : r \leq \lambda \leq 1, x_0 \in C_0, x_1 \in C_1\}.$$ 

To obtain the conclusion one must check that the sets $C = \overline{\text{aff}}(A)$, $C_0$ and $C_1$ satisfy the conditions in lemma 1 to apply it.

$C$ and $C_1$ are clearly bounded, $\sigma(X, F)$-closed and convex. $K_\varepsilon$ is $\| \cdot \|$-compact, hence $C_0$ is $\sigma(X, F)$-closed. Since $K_\varepsilon$ is convex it is easy to see that $C_0$ is also convex.

1. $C_0 \subset \overline{\text{aff}}(A)$; since $K_\varepsilon$ can be covered by finitely many balls of arbitrary small radius, it is not difficult to check that $\alpha(C_0) \leq 2 \varepsilon$.

2. $\overline{\text{aff}}(A)$ is not a subset of $C_1$, (since $x \notin C_1$).

3. $\overline{\text{aff}}(A) = \overline{\text{aff}}(C_1 \cup C_0)$. To check it we show that $\text{co} (A) \subset \text{co} (C_1 \cup C_0)$. To do so, set $B_1 = \text{co} (A \cap H)$ and $B_2 = \text{co} (A \setminus H)$. It is clear that $\text{co} (A) \subset \text{co} (B_1 \cup B_2)$. Now since $A \cap H \subset C_0$ one must have $C_0 \supset B_1$ and clearly $B_2 \subset C_1$.

Since $\alpha(C_0) \leq 2 \varepsilon$, we have $\alpha(C_0) < 3 \varepsilon$ and we can take $\varepsilon' = \frac{3 \varepsilon}{4}$ in lemma 1 and then it will be enough to take $r < \frac{\varepsilon}{3M}$. Now the lemma applies to give the conclusion for i) and ii). Property iii) easily follows from the definition of the set $D_r := D_r(x)$ and the fact that $g(y) \leq \eta$ for $y \in C_1$.

We shall iterate now the former lemma to be able to ensure that $\varepsilon$-quasi-denting points for an arbitrary subset $B$ should be $3\varepsilon$-quasi-denting points in some convex set of a sequence $\{B_n\}$ associated to $B$.

Lemma 5. (Iteration lemma) Let $B$ be a bounded subset of the normed space $X, F \subset X^*$ 1-norming such that for some $\varepsilon > 0$ fixed, every $x \in B$ is an $\varepsilon$-$\sigma(X, F)$-quasi-denting point for $B$. Then there is a sequence

$$B_0 = \overline{\text{aff}}(B) \supseteq B_1 \supseteq B_2 \supseteq \ldots \supseteq B_n \supseteq \ldots$$

of convex, $\sigma(X, F)$-closed subsets of $\overline{\text{aff}}(B)$ such that for every $x \in B$ there exists $p \geq 0$ satisfying $x \in B_p$ and $x$ is a $3\varepsilon$-$\sigma(X, F)$-quasi-denting point for $B_p$.

Proof. We shall construct by recurrence sequences of sets

$$B_0 = \overline{\text{aff}}(B) \supseteq B_1 \supseteq B_2 \supseteq \ldots \supseteq B_n \supseteq \ldots$$

and $B := \bar{B}_0, \bar{B}_1, \ldots, \bar{B}_n, \ldots$ such that

$$B_n := \overline{\text{aff}} \left( B \cap \bar{B}_1 \cap \bar{B}_2 \cap \ldots \cap \bar{B}_n \right) \text{ if } B \cap \bar{B}_1 \cap \bar{B}_2 \cap \ldots \cap \bar{B}_n \neq \emptyset$$

and given \( x \in B \), if \( x \in \left( B \cap \hat{B}_1 \cap \hat{B}_2 \cap \ldots \cap \hat{B}_{n-1} \right) \setminus \hat{B}_n \) then \( x \) is a \( 3\varepsilon \sigma(X,F) \)-quasi-denting point for

\[
B_{n-1} = \overline{\sigma} \left( B \cap \hat{B}_1 \cap \hat{B}_2 \cap \ldots \cap \hat{B}_{n-1} \right).
\]

Indeed, set \( B_0 := \overline{\sigma}(B) \) and \( \hat{B}_0 := B \). Now fix \( x \in B \), by hypothesis we fix \( g_x \in F, \eta_x \in \mathbb{R} \) such that the half space \( H_x = \{ y \in X : g_x(y) > \eta_x \} \) satisfies

\[
x \in H_x \cap B \text{ and } \alpha(H_x \cap B) < \varepsilon.
\]

Let \( M = \text{diam} \left( B_0 \right) \). At each point \( x \) from \( B \) together with the corresponding \( H_x \in \mathcal{H}(F) \), we may apply the former lemma to obtain \( D^1_r(x) \), \( \sigma(X,F) \)-closed and convex and \( r \in [0,1] \) with the properties described in lemma 4. Now define

\[
\hat{B}_1 := \bigcap_{x \in B_0} D^1_r(x).
\]

Note that if \( x \in B \setminus \hat{B}_1 \) then, there exists \( y \in B \) such that \( x \in \overline{\sigma}(B) \setminus D^1_r(y) \) and \( x \) is a \( 3\varepsilon \sigma(X,F) \)-quasi-denting point for \( B_0 = \overline{\sigma}(B) \).

Notice that if \( B \cap \hat{B}_1 = \emptyset \) we would have finished the proof since every \( x \in B \) would be a \( 3\varepsilon \sigma(X,F) \)-quasi-denting point for \( B_0 \). So assume \( B \cap \hat{B}_1 \neq \emptyset \), we shall define a set \( B_1 \) as

\[
B_1 := \overline{\sigma}(B \cap \hat{B}_1).
\]

Consider the set \( B \cap \hat{B}_1 \) and \( H_x \) at every point \( x \in B \cap \hat{B}_1 \). Since

\[
diam \left( \overline{\sigma}(B \cap \hat{B}_1) \right) \leq M \text{ and } \alpha(H_x \cap B \cap \hat{B}_1) < \varepsilon
\]

we apply lemma 4 to the set \( B \cap \hat{B}_1 \) and this time we will obtain sets \( D^2_r(x) \) with the properties given by the lemma and \( r \) being the same \( r \) as above. Now define

\[
\hat{B}_2 := \bigcap_{x \in B \cap \hat{B}_1} D^2_r(x)
\]

As we did before, if \( x \in (B \cap \hat{B}_1) \setminus \hat{B}_2 \) there must be \( y \in B \cap \hat{B}_1 \) such that

\[
x \in B_1 = \overline{\sigma}(B \cap \hat{B}_1) \setminus D^2_r(y)
\]

and \( x \) is a \( 3\varepsilon \sigma(X,F) \)-quasi-denting point for \( B_1 \). It follows now by recurrence that such sequences can be built and it will be finite if

\[
B \cap \hat{B}_1 \cap \hat{B}_2 \cap \ldots \cap \hat{B}_n = \emptyset
\]

To finish the proof we need to show that for every \( x \in B \) there exists \( p \geq 0 \) such that \( x \in \left( B \cap \ldots \cap \hat{B}_{p-1} \right) \setminus \hat{B}_p \). So suppose this is not the case, i.e., there exists \( x \in B \) (which will be fixed from now on), such that \( x \in B \cap \hat{B}_1 \cap \ldots \cap \hat{B}_p \) for every \( p = 1, 2, \ldots \). Let us consider the sets \( D^p_r(x) \) defined for the point \( x, g_x \) and \( \eta_x \) at each step \( p = 1, 2, \ldots \). Recall from lemma 4 that

\[
\sup g_x \left( D^1_r(x) \right) \leq (1 - r) \sup g_x(B_0) + r \eta_x.
\]
So for \( p = 2 \), and bearing in mind that \( B_1 \subset \mathcal{D}^1_\varepsilon(x) \) we have
\[
\sup g_x(D^2(x)) \leq (1 - r) \sup g_x(B_1) + r \varepsilon
\]
\[
\leq (1 - r) \sup g_x(D^1_\varepsilon(x)) + r \varepsilon \leq (1 - r) [(1 - r) \sup g_x(B_0) + r \varepsilon] + r \varepsilon
\]
\[
= (1 - r)^2 \sup g_x(B_0) + r \varepsilon \leq 1
\]
Now by induction we should have
\[
\sup g_x(D^n_\varepsilon(x)) \leq (1 - r)^n \sup g_x(B_0) + r \varepsilon [1 + (1 - r) + \ldots + (1 - r)^{n-1}]
\]
\[
= (1 - r)^n \sup g_x(B_0) + \varepsilon (1 - (1 - r)^n) = \varepsilon (1 - (1 - r)^n) (\sup g_x(B_0) - \varepsilon)
\]
for every integer \( n \) such that \( x \in B \cap \tilde{B}_1 \cap \ldots \cap \tilde{B}_{n-1} \).

Now since \( (1 - r)^n \) tends to 0 as \( n \) goes to infinity and \( \varepsilon \) is chosen small enough so that \( \sup g_x(D^n_\varepsilon(x)) < g(x) \) one can choose \( n \) large enough so that \( \sup g_x(D^n_\varepsilon(x)) < g(x) \) which is a contradiction with assuming \( x \in D^n_\varepsilon(x) \).

Thus, there exists \( n \in \mathbb{N} \) such that \( x \in B \cap \tilde{B}_1 \cap \ldots \cap \tilde{B}_{n-1} \) and \( x \notin D^n_\varepsilon(x) \) hence \( x \in (B \cap \tilde{B}_1 \cap \ldots \cap \tilde{B}_{n-1}) \setminus \tilde{B}_n \) as we wanted. \( \square \)

The step connecting Kuratowski’s index with dentability follows now from theorem 6:

**Corollary 12.** Let \( B \) be a bounded subset of the normed space \( X, F \subset X^* \) 1-norming subspace for it such that for some \( \varepsilon > 0 \) fixed, every \( x \in B \) is an \( \varepsilon \)-\( \sigma(X,F) \)-quasi-denting point for \( B \). Then there is a countable family \( \{T_n : n = 1, 2, \ldots \} \) of \( \sigma(X,F) \)-closed and convex subsets of \( \overline{\sigma(B)} \) such that for every \( x \in B \) there exists \( p > 0 \) such that \( x \in T_p \) and there is \( H \in \mathbb{H}(F) \) with \( x \in H \) and \( \text{diam}(H \cap T_p) < \varepsilon \).

**Proof.** If we set \( B_0 \supset B_1 \supset \ldots \supset B_n \supset \ldots \) as in lemma 5, we know that
\[
\delta_F(3\varepsilon - \sigma(X,F) - \text{quasi-denting points of } B_p, B_p, 3\varepsilon < \eta_{3\varepsilon} < \omega_1
\]
and therefore the family of derived sets \( \{D^\beta_{3\varepsilon,F}(B_p) : \beta < \eta_{3\varepsilon}, p = 1, 2, \ldots \} \), provides us a countable family \( \{T_n : n = 1, 2, \ldots \} \) with the required properties. \( \square \)

Now we arrive to the proof of theorem 11

(1)\( \Rightarrow \) (2) Follows from theorem 2 in the introduction.

(2)\( \Rightarrow \) (1) It is clear that condition (2) must be true for any equivalent norm and it is not a restriction to assume that the given norm is \( \| \cdot \| \) making \( F \) a 1-norming subspace for it. Then we have the conditions of corollary 12 for every set
\[
X_{n,\varepsilon} \cap B(0, m) = n = 1, 2, \ldots, m = 1, 2, \ldots
\]
and we will have countable families \( \{T^n_{p,m,\varepsilon} : p = 1, 2, \ldots \} \), \( n = 1, 2, \ldots, m = 1, 2, \ldots \) such that for every \( x \in X_{n,\varepsilon} \cap B(0, m) \) there is \( p \geq 0 \) such that
\[
x \in T^n_{p,m,\varepsilon} \text{ and there is } H \in \mathbb{H}(F) \text{ with } x \in H \text{ and } \text{diam}(H \cap T^n_{p,m,\varepsilon}) < 3\varepsilon
\]
If we set
\[
Y^n_{p,m,\varepsilon} := \{x \in T^n_{p,m,\varepsilon} : \text{there is } H \in \mathbb{H}(F), x \in H \text{ and } \text{diam}(H \cap T^n_{p,m,\varepsilon}) < 3\varepsilon\}
we have $X = \cup \{ Y^n_{p,m} : n, m, p = 1, 2, \ldots \}$ and we have the decomposition fixed in theorem 2 which is equivalent to have a $\sigma(X, F)$-lower semi-continuous LUR norm on $X$.

(2)$\Rightarrow$(3) Is obvious.

(3)$\Rightarrow$(2) Given $x \in X \setminus \{ 0 \}$ let $r(x) > 0$ such that $r(x)x \in A$. By hypothesis, for every $k \in \mathbb{N}$, $A = \cup_n A_{n,k}$ with the property that for every $x \in A_{n,k}$ there exists $H \in \mathbb{H}(F), x \in H$ such that $\alpha (A_{n,k} \cap H) < \frac{1}{k}$. For $q \in \mathbb{Q}, n, m, k \in \mathbb{N}$ define

$$A_{n,k}^{q,m} := \{ y \in X \setminus \{ 0 \} : r(y)y \in A_{n,k}, 0 < \frac{1}{q} - \frac{1}{m} < \frac{1}{r(y)} < \frac{1}{q} \}.$$ 

We shall show that $X \setminus \{ 0 \} = \cup \{ A_{n,k}^{q,m} : n, m, k \in \mathbb{N}, q \in \mathbb{Q} \}$ and for every $\varepsilon > 0$, and $x \in X \setminus \{ 0 \}$ there exist $n, m, k \in \mathbb{N}, q \in \mathbb{Q}, H \in \mathbb{H}(F)$ with $x \in H$ such that $\alpha (A_{n,k}^{q,m} \cap H) < \varepsilon$.

So, given $\varepsilon > 0$ and $x_0 \in X \setminus \{ 0 \}$, consider $r(x_0) > 0$ and let $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{r(x_0)}{2 \varepsilon}$. For this $k$ (fixed), let $n \in \mathbb{N}$ such that $r(x_0)x_0 \in A_{n,k}$. By the property of $A$, there exist $f \in F$ and $\mu \in \mathbb{R}$ such that

$$r(x_0)x_0 \in H = \{ x \in X : f(x) > \mu \} \text{ and } \alpha (A_{n,k} \cap H) < \frac{1}{k}.$$ 

Therefore, there are sets $B_i, i = 1, \ldots, j$ with $\text{diam} (B_i) < \frac{1}{k}$ such that

$$A_{n,k} \cap H \subset \bigcup_{i=1}^{j} B_i.$$ 

For every $i \in \{ 1, \ldots, j \}$ fix $x_i \in B_i$. Take $m \in \mathbb{N}$ such that $m > \frac{2M}{\varepsilon} + r(x_0)$ and let $M = \max_i \{ \| x_i \| \}$. Finally let $q \in \mathbb{Q}$ such that

$$\frac{1}{q} - \frac{1}{m} < \frac{1}{r(x_0)} < \frac{1}{q} \text{ and } f(x_0) > \frac{\mu}{q} > \frac{\mu}{r(x_0)}.$$ 

Take now the $\sigma(X, F)$-open half space $H' := \{ x \in X : f(x) > \frac{\mu}{q} \}$. It is clear that $x_0 \in A_{n,k}^{q,m} \cap H'$. Let, for every $i \in \{ 1, \ldots, j \}, u_i = \frac{1}{r(x_0)}x_i$. Let us prove that $\alpha (A_{n,k}^{q,m} \cap H') < 2\varepsilon$ by checking $A_{n,k}^{q,m} \cap H' \subset \bigcup_{i=1}^{j} B(u_i; \varepsilon)$. To do so take any $y \in A_{n,k}^{q,m} \cap H'$. In particular $f(y) > \frac{\mu}{q}$, hence

$$f(r(y)y) = r(y)f(y) > r(y)\frac{\mu}{q} > \mu.$$ 

Therefore, $r(y)y \in A_{n,k} \cap H$. So, there must be $x_i$, for some $i \in \{ 1, \ldots, j \}$ such that $\| r(y)y - x_i \| < \frac{1}{k}$ thus, $\| y - \frac{1}{r(y)}x_i \| < \frac{1}{r(y)}$. So,

$$\| y - u_i \| = \| y - \frac{1}{r(x_0)}x_i \| \leq \| y - \frac{1}{r(y)}x_i \| + \| \frac{1}{r(y)}x_i - \frac{1}{x_0}x_i \| <$$

$$< \frac{1}{k} \frac{1}{r(y)} + \| x_i \| \left( \frac{1}{r(y)} - \frac{1}{r(x_0)} \right) \leq \frac{1}{k} \left( \frac{1}{r(x_0)} + \frac{1}{m} \right) + M \frac{1}{m} < \varepsilon$$

$\square$
In order to give our last result we should introduce some terminology. Recall that in a topological space $X$ a family of subsets of $X$, $\mathcal{A}$, is said to be relativelocally finite (resp. isolated) if for every $x \in \bigcup \{A : A \in \mathcal{A}\}$ there exists an open set $V \ni x$ such that the set $\{A : A \in \mathcal{A}, A \cap V \neq \emptyset\}$ is finite (resp. contains exactly one element). If $P$ is any of the properties above, as usual, the family $\mathcal{A}$ is said to be $\sigma$-$P$ if $\mathcal{A} = \bigcup \{\mathcal{A}_n : n \in \mathbb{N}\}$ in such a way that for each $n \in \mathbb{N}$ the family $\mathcal{A}_n$ has property $P$.

When dealing with a normed space $X$ and $F \subset X^*$ norming, we shall talk of slicely norm whenever the open set $V$ can be chosen to be an open half space from $\mathbb{H}(F)$.

Finally a network for a topological space $X$ is a collection $\mathcal{N}$ of subsets of $X$ such that whenever $x \in U$ with $U$ open, there exists $N \in \mathcal{N}$ with $x \in N \subset U$.

Recall that from [13] it follows that given a Banach space $X$ and a norming subspace for it, $F$, $X$ admits a $\sigma(X, F)$-lower semi-continuous LUR equivalent norm if, and only if, the norm topology has a $\sigma$-slicely isolated network, see also [6, 15, 16].

**Corollary 13.** Let $(X, \| \cdot \|)$ be a normed space and $F \subset X^*$ norming. The following conditions are equivalent:

1. The norm topology admits a $\sigma$-slicely relatively locally finite network;
2. $X$ admits an equivalent $\sigma(X, F)$-lower semi-continuous LUR norm.

**Proof.** By the result in [13] mentioned above we only have to show that (1)$\Rightarrow$(2) and this will be done through the equivalent conditions in theorem 11. To do so, one may assume that the network $\mathcal{N} = \bigcup \{\mathcal{N}_n : n \in \mathbb{N}\}$ satisfying (1) is such that for each $n \in \mathbb{N}$ the family $\mathcal{N}_n$ consists of pairwise disjoint sets. Indeed, if this is not the case then for each $n, m \in \mathbb{N}$ we define the family

$$\mathcal{N}_n^m := \{N_1 \cap \ldots \cap N_m : N_i \in \mathcal{N}_n, i = 1, 2, \ldots, m\}$$

and the sets $S_n^m := \{x \in X : x \in A \in \mathcal{N}_n^m \text{ and } \text{ord}(x, \mathcal{N}_n) = m\}$. Now we set

$$\mathcal{N}_n^m \cap S_n^m := \{A \cap S_n^m : A \in \mathcal{N}_n^m\}$$

It is easy to show that for each $n, m \in \mathbb{N}$ the sets in this family are pairwise disjoint, $\cup \{\mathcal{N}_n^m \cap S_n^m : m \in \mathbb{N}\}$ is a refinement for $\mathcal{N}_n$ and $\{\mathcal{N}_n^m \cap S_n^m : n, m \in \mathbb{N}\}$ is network for the norm topology which is $\sigma$-slicely relatively locally finite.

Now fix $\varepsilon > 0$. For every positive integer $n$ define

$$X_{n, \varepsilon} := \{x \in \bigcup \{N : N \in \mathcal{N}_n\} \text{ such that } x \in N \subset B(x; \varepsilon)\} = \{x \in X : \text{there exists } N \in \mathcal{N}_n \text{ with } x \in N \subset B(x; \varepsilon)\}$$

Since $\mathcal{N}$ is a network for the norm topology we have $X = \bigcup \{X_{n, \varepsilon} : n \in \mathbb{N}\}$. Fix $x \in X_{n, \varepsilon}$. Since the network is $\sigma$-slicely relatively locally finite, there must be $H \in \mathbb{H}(F)$ such that $x \in H$ and $H \cap \bigcup \{N : N \in \mathcal{N}_n\} = H \cap N_1 \cup \ldots \cup H \cap N_p$ for a finite number of sets $N_i \in \mathcal{N}_n$.

If we consider $y \in H \cap X_{n, \varepsilon}$ we have $y \in H \cap N_j$ for some $j \in \{1, 2, \ldots, p\}$, and by the very definition of $X_{n, \varepsilon}$ and the disjointness of the family $\mathcal{N}_n$, $y \in N_j \subset B(y; \varepsilon)$. So for $N_j$ we have diam $(N_j) < 2\varepsilon$.

Therefore we have $\{p_1, p_2, \ldots, p_q\} \subset \{1, 2, \ldots, p\}$ so that $H \cap X_{n, \varepsilon} \subset N_{p_1} \cup \ldots \cup N_{p_q}$ and diam $(N_{p_i}) < 2\varepsilon$. So $\alpha(H \cap X_{n, \varepsilon}) < 2\varepsilon$ and the proof is done. □
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