THE UNIT BALL OF THE HILBERT SPACE IN ITS WEAK TOPOLOGY

ANTONIO AVILÉS

Abstract. We show that the unit ball of \( \ell_p(\Gamma) \) in its weak topology is a continuous image of \( \sigma_1(\Gamma)^\mathbb{N} \) and we deduce some combinatorial properties of its lattice of open sets which are not shared by the balls of other equivalent norms when \( \Gamma \) is uncountable.

For a set \( \Gamma \) and a real number \( 1 < p < \infty \), the Banach space \( \ell_p(\Gamma) \) is a reflexive space, hence its unit ball is compact in the weak topology and in fact, it is homeomorphic to the following closed subset of the Tychonoff cube \([-1, 1]^\Gamma\):

\[
B(\Gamma) = \left\{ x \in [-1, 1]^\Gamma : \sum_{\gamma \in \Gamma} |x_\gamma| \leq 1 \right\}.
\]

The homeomorphism \( h : B(\ell_p(\Gamma)) \to B(\Gamma) \) is given by \( h(x)_\gamma = \text{sign}(x_\gamma) \cdot |x_\gamma|^p \).

The spaces homeomorphic to closed subsets of some \( B(\Gamma) \) constitute the class of uniform Eberlein compacta, introduced by Benyamini and Starbird [6]. The compact subset \( \sigma_k(\Gamma) \), the compact subset of \( \{0, 1\}^\Gamma \) which consists of the functions with at most \( k \) nonzero coordinates (\( k \) a positive integer) is an example of a uniform Eberlein compact. In fact, the following result was proven in [5]:

**Theorem 1** (Benyamini, Rudin, Wage). Every uniform Eberlein compact of weight \( \kappa \) is a continuous image of a closed subset of \( \sigma_1(\kappa)^\mathbb{N} \).

In the same paper [5], it was posed the problem whether in fact, it was possible to get any uniform Eberlein compact as a continuous image of the full \( \sigma_1(\Gamma)^\mathbb{N} \). This question was answered in the negative by Bell [2], by considering the following property:

A compact space \( K \) verifies property \( (Q) \) if for every uncountable regular cardinal \( \lambda \) and every family \( \{U_\alpha, V_\alpha\}_{\alpha < \lambda} \) of open subsets of \( K \) with \( U_\alpha \subset V_\alpha \) one of the following two alternatives must hold:

1. either there exists a set \( A \subset \lambda \) with \( |A| = \lambda \) such that \( U_\alpha \cap U_\beta = \emptyset \) for every two different elements \( \alpha \) and \( \beta \) in \( A \),

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(2) or either there exists a set \( A \subset \lambda \) with \(|A| = \lambda\) such that \( V_{\alpha} \cap V_{\beta} \neq \emptyset \) for every two different elements \( \alpha \) and \( \beta \) in \( A \).

Bell proved in [2] that property (Q) is satisfied by all polyadic spaces, that is, continuous images of \( \sigma_1(\Gamma)^{\Lambda} \) for any sets \( \Gamma \) and \( \Lambda \), (this concept was introduced in [8] and studied earlier by Gerlits [7]), but he constructed a uniform Eberlein compact without property (Q). Later, Bell [4] provided another example of a uniform Eberlein compact which is not a continuous image of any \( \sigma_1(\Gamma)^{N} \) but which is nevertheless polyadic. Our main result is the following:

**Theorem 2.** \( B(\Gamma) \) is a continuous image of \( \sigma_1(\Gamma)^{N} \).

As a consequence, \( B(\Gamma) \) satisfies property (Q) as well as other properties of the same type introduced by Bell in [4] and [3]. However, if \( \Gamma \) is uncountable, we show in Theorem 4 that a modification of one of the examples of Bell provides an equivalent norm on \( \ell_p(\Gamma) \) whose unit ball is not a continuous image of \( \sigma_1(\Gamma)^{N} \), indeed not satisfying property (Q). In particular, we are showing the existence of equivalent norms in the nonseparable \( \ell_p(\Gamma) \) whose closed unit balls are not homeomorphic in the weak topology. This contrasts with the separable case, since the balls of all separable reflexive Banach spaces are weakly homeomorphic [1, Theorem 1.1]. We refer to [1] for information about the problem whether the balls of equivalent norms in a Banach space are weakly homeomorphic in the separable case.

**Proof of Theorem 2:** For a set \( \Delta \) we will use the notation \( B^+(\Delta) = B(\Delta) \cap [0, 1]^\Delta \). First, we point out that \( B(\Gamma) \) is a continuous image of \( B^+(\Gamma) \). Indeed, if we consider \( \Gamma^0 = \Gamma \times \{a, b\} \), we have a continuous surjection \( \psi : B^+(\Gamma^0) \to B(\Gamma) \) given by \( \psi(x)_{\gamma} = x_{(\gamma, a)} - x_{(\gamma, b)} \).

In a second step, we apply the standard procedure to express the space \( B^+(\Gamma) \) as a continuous image of a totally disconnected compact \( L_0 \). We fix a sequence \( (r_n)_{n=0}^{\infty} \) of positive real numbers such that \( \sum_{n=0}^{\infty} r_n = 1 \) and such that the continuous map \( \phi : [0, 1]^\Gamma \to [0, 1]^\Gamma \) given by \( \phi(x) = \sum_{n=0}^{\infty} r_n x_n \) is surjective, for example \( r_n = \frac{1}{2^n \pi^2} \). We consider the power \( \phi^F : [0, 1]^{\Gamma \times \mathbb{N}} \to [0, 1]^\Gamma \) and then we set:

\[
L_0 = (\phi^F)^{-1}(B^+(\Gamma)), \quad f = \phi^F|_{L_0},
\]

so that \( f : L_0 \to B^+(\Gamma) \) is a continuous surjection. It will be convenient to have an explicit description of \( L_0 \). For \( x \in [0, 1]^{\Gamma \times \mathbb{N}} \) and \( n \in \mathbb{N} \), we define \( N_n(x) = |\{ \gamma \in \Gamma : x_{(\gamma, n)} = 1 \}|.\)
Let $M\subseteq \sigma$ be some notation. An element of $\sigma$ is all $\gamma$ closed in the integer part of the expression of $\sigma$. All subsets of $N$ form a neighborhood which separates $Z$, existence of such a function follows the fact that any countable product of spaces is a continuous image of $\sigma$. On the other hand, for any metrizable compact, it is a continuous image of $\sigma_i$. Namely, if $\tau_n \leq \sigma_n$ for all $n \in \mathbb{N}$, then $\tau \in Z$. Associated to such a set $\tau_n$ we construct the following space:

$$
\mathcal{K}(Z, \Gamma) = \{x \in \{0, 1\}^{\Gamma \times \mathbb{N}} : (N_n(x))_{n \in \mathbb{N}} \in Z\}.
$$

We have that $L_0 = \mathcal{K}(Z_0, \Gamma)$ where $Z_0 = \{s \in \mathbb{N}^\Gamma : \sum_{i \in \mathbb{N}} r_i s_i \leq 1\}$. Note that $Z_0$ is indeed compact since it is a closed subset of $\prod_{n \in \mathbb{N}} \{0, \ldots, M_n\}$ where $M_n$ is the integer part of $\frac{1}{r_s}$. The proof will be complete after the following lemma:

**Lemma 3.** Let $Z$ be a compact subset of $\mathbb{N}^\Gamma$ such that if $\sigma \in Z$ and $\tau_n \leq \sigma_n$ for all $n \in \mathbb{N}$, then $\tau \in Z$. Then $\mathcal{K}(Z, \Gamma)$ is a continuous image of $\sigma_1(\Gamma)^\mathbb{N}$.

**PROOF:** First we check that $\mathcal{K}(Z, \Gamma)$ is a closed subset of $\{0, 1\}^{\Gamma \times \mathbb{N}}$ and hence compact. Namely, if $x \in \{0, 1\}^{\Gamma \times \mathbb{N}} \setminus \mathcal{K}(Z, \Gamma)$, then $(N_n(x))_{n \in \mathbb{N}} \notin Z$ and since $Z$ is closed in $\mathbb{N}^\Gamma$, there is a finite set $F \subset \mathbb{N}$ such that $\sigma \notin Z$ whenever $\sigma_n = N_n(x)$ for all $n \in F$. Indeed, by the definition of $Z$, if $\tau \in \mathbb{N}^\Gamma$ and $\tau_n \geq \sigma_n$ of all $n \in F$, also $\tau \notin Z$. In this case,

$$
W = \{y \in \{0, 1\}^{\Gamma \times \mathbb{N}} : y_{\gamma,n} = 1 \text{ whenever } n \in F \text{ and } x_{\gamma,n} = 1\}
$$

is a neighborhood which separates $x$ from $\mathcal{K}(Z, \Gamma)$ and this finishes the proof that $\mathcal{K}(Z, \Gamma)$ is closed. Since $Z$ is compact, for every $n \in \mathbb{N}$ there exists $M_n \in \mathbb{N}$ such that $\sigma_n \leq M_n$ for all $\sigma \in Z$. We define the following compact space:

$$
L_1 = Z \times \prod_{m \in \mathbb{N}} \prod_{i=0}^{M_m} \sigma_i(\Gamma)
$$

Note that $L_1$ is a continuous image of $\sigma_1(\Gamma)^\mathbb{N}$. On the one hand, since $Z$ is a metrizable compact, it is a continuous image of $\{0, 1\}^\mathbb{N}$ and in particular of $\sigma_1(\Gamma)^\mathbb{N}$. On the other hand, for any $i \in \mathbb{N}$, the space $\sigma_i(\Gamma)$ can be viewed as the family of all subsets of $\Gamma$ of cardinality at most $i$. In this way, we consider the continuous surjection $p : \sigma_1(\Gamma)^i \twoheadrightarrow \sigma_i(\Gamma)$ given by $p(x_1, \ldots, x_i) = x_1 \cup \cdots \cup x_i$. From the existence of such a function follows the fact that any countable product of spaces $\sigma_i(\Gamma)$ is a continuous image of $\sigma_1(\Gamma)^\mathbb{N}$, and in particular, the second factor in the expression of $L_1$ is such an image.

It remains to define a continuous surjection $g : L_1 \twoheadrightarrow \mathcal{K}(Z, \Gamma)$. We first fix some notation. An element of $L_1$ will be written as $(z, x)$ where $z \in Z$ and $x \in \mathbb{N}^\Gamma$.
\( \prod_{m \in \mathbb{N}} \prod_{i=0}^{M_m} \sigma_i(\Gamma). \) At the same time, such an \( x \) is of the form \((x^m)_{m \in \mathbb{N}}\) with \(x^m \in \prod_{i=0}^{M_m} \sigma_i(\Gamma)\) and again each \(x^m\) is \((x^m)_{\gamma}^{\Gamma}\) where \((x^m)_{\gamma}^{\Gamma} \in \sigma_1(\Gamma)\). Finally \(x_{\gamma}^{m,i} = (x^m)^{\Gamma}_{\gamma} \in \sigma_1(\Gamma) \subset \{0, 1\}^{\Gamma}. \) The function \(g : L_1 \rightarrow K(Z, \Gamma) \subset \{0, 1\}^{\Gamma \times \mathbb{N}}\) is defined as follows:

\[
g(z, x)_{\gamma,m} = x_{\gamma}^{m,z(m)}
\]

Observe that \(g(x, z)\) maps indeed \(L_1\) onto \(K(Z, \Gamma)\) because for every \(m\), \((x_{\gamma}^{m,z(m)})_{\gamma \in \Gamma}\) is an arbitrary element of \(\sigma_{\alpha(m)}(\Gamma)\).

**Theorem 4.** Let \(\Gamma\) be an uncountable set and \(1 < p < \infty\). There exists an equivalent norm on \(\ell_p(\Gamma)\) whose unit ball does not satisfy property \((Q)\) and hence it is not polyadic.

**PROOF:** This is a variation of an example of Bell [2], originally a scattered compact, so that to make it absolutely convex. We will consider \(\omega_1\) as a subset of \(\Gamma\). Let \(\phi : \omega_1 \rightarrow \mathbb{R}\) be a one-to-one map and

\[
G = \{(\alpha, \beta) \in \omega_1 \times \omega_1: \phi(\alpha) < \phi(\beta) \iff \alpha \leq \beta\}.
\]

We define an equivalent norm on \(\ell_p(\Gamma) \times \ell_p(\Gamma) \sim \ell_p(\Gamma)\) by

\[
\|X\| = \sup\{|x|_p, |y|_p, |x_\alpha| + |y_\beta|: (\alpha, \beta) \in G\}
\]

and let \(K\) be its unit ball considered in its weak topology. Fix numbers \(1 < \xi_1 < \xi_2 < 2^{1/3}\). The families of open sets

\[
U_\alpha = \{(x, y) \in K: |x_\alpha| + |y_\alpha| > \xi_1\}, \quad \alpha < \omega_1
\]

\[
V_\alpha = \{(x, y) \in K: |x_\alpha| + |y_\alpha| > \xi_2\}, \quad \alpha < \omega_1
\]

verify that \(U_\alpha \subset V_\alpha\) and that for any \(\alpha, \beta < \omega_1\), \(U_\alpha \cap U_\beta = \emptyset\) if and only if \((\alpha, \beta) \in G\) if and only if \(V_\alpha \cap V_\beta = \emptyset\). Namely, if there is some \((x, y) \in V_\alpha \cap V_\beta\), then

\[
|x_\alpha| + |y_\alpha| + |x_\beta| + |y_\beta| > \xi_1 + \xi_1 > 2
\]

and therefore either \(|x_\alpha| + |x_\beta| > 1\) or \(|y_\alpha| + |y_\beta| > 1\) and this implies that \((\alpha, \beta) \notin G\) since \((x, y) \in K\). On the other hand, if \((\alpha, \beta) \notin G\) then the element \((x, y) \in \ell_p(\Gamma) \times \ell_p(\Gamma)\) which has all coordinates zero except \(x_\alpha = x_\beta = y_\alpha = y_\beta = 2^{-\frac{1}{p}}\) lies in \(U_\alpha \cap U_\beta\). Since there is no uncountable well ordered (or inversely well ordered) subset of \(\mathbb{R}\) there is no uncountable subset of \(\omega_1\) such that \(A \times A \subset G\) or \((A \times A) \cap G = \emptyset\). Therefore, the families \(\{U_\alpha\}\) and \(\{V_\alpha\}\) witness the fact that \(K\) does not have property \((Q)\) and hence, it is not polyadic.

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Departamento de Matemáticas, Universidad de Murcia, 30100 Espinardo (Murcia), Spain
E-mail address: avileslo@um.es