, ( ), 1–23 ( )

Boolean Metric Spaces and Boolean Algebraic Varieties

Antonio Avilés
avileslo@um.es

Departamento de Matemáticas. Universidad de Murcia, Spain

Abstract

The concepts of Boolean metric space and convex combination are used to characterize polynomial maps $A^n \longrightarrow A^m$ in a class of commutative Von Neumann regular rings including p-rings, that we have called CFG-rings. In those rings, the study of the category of algebraic varieties (i.e. sets of solutions to a finite number of polynomial equations with polynomial maps as morphisms) is equivalent to the study of a class of Boolean metric spaces, that we call here CFG-spaces.

*MSC Primary: 13A. MSC Secondary: 06E30, 16E50, 51F99.
[1]Author supported by FPU grant of SEEU-MECD, Spain.

Notations and conventions

Throughout this work, $(B,+,\cdot)$ will be a Boolean ring where the operation $a \vee b = a+b+ab$ is the analogue for set union, the order $a \leq b \Leftrightarrow ab = a$ is the analogue for set inclusion and for each $a \in B$, $\bar{a} = a+1$ is the analogue for the set complement of a.

All rings will be commutative with identity. Regular ring will mean here commutative Von Neumann regular ring, i.e. a (commutative) ring for which any principal ideal is generated by an idempotent, also known as absolutely flat rings, see [6], [12]. Unless otherwise stated, A will be a regular ring, $B(A)$ will denote the set of the idempotent elements of A and $e : A \longrightarrow B(A)$ will be the map that sends each $a \in A$ to the only idempotent $e(a) \in B(A)$ such that $aA = e(a)A$. The set $B(A)$ has a structure of Boolean ring with product inherited from A and with the sum $a \tilde{+} b = (a-b)^2$. For $a_1, \ldots, a_n \in B(A)$ with $a_i a_j = 0$ for $i \neq j$, it holds $a = a_1 + \cdots + a_n = a_1 \tilde{+} \cdots \tilde{+} a_n = a_1 \vee \cdots \vee a_n$. In this case we will denote a by $a_1 \oplus \cdots \oplus a_n$.

Given a prime $p \in \mathbf{Z}$, a p-ring is a ring A for which $px = 0$ and $x^p = x$ for all $x \in A$. In particular, a Boolean ring is a 2-ring. Any p-ring is a regular ring with $e(x) = x^{p-1}$.

An algebraic variety over a ring A is a set $U \subset A^n$ which is the set of solutions to a finite number of polynomial equations. If $U \subset A^n$ and $V \subset A^m$ are algebraic varieties, a map $f : U \longrightarrow V$ is called a polynomial map if there are polynomials $f_1, \ldots, f_m \in A[X_1, \ldots X_n]$ such that $f(x) = (f_1(x), \ldots, f_m(x))$. When $A = B$ is a Boolean ring the usual terms are Boolean domain and Boolean transformation, see [13] and [14].

Introduction

Boolean metric spaces (Definition 1.1) appeared in several works in the 1950's and 1960's [2], [3], [4], [5], [7] and [8], where some authors investigated the analogue for some topics in Geometry such as *betweeness*, motions or topology in some of those spaces, as Boolean algebras and some rings where a suitable Boolean metric could be defined. In some papers, [1], [9], [11] and [15], a special attention was paid to p-rings, that admit a metric space structure over its ring of idempotents. In fact, if A is a regular ring, then A^n is a Boolean metric space over $B(A)$ with the distance

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = e(x_1 - y_1) \vee \cdots \vee e(x_n - y_n).$$

We will show the close relation that exists between the theory of Boolean metric spaces and the Algebraic Geometry over CFG-rings. We define a regular ring A to be an CFG-ring if there are $x_1, \ldots, x_n$ in A such that any element in A is of the form $\sum a_i x_i$ where $a_1, \ldots a_n \in B(A)$ and $a_1 \oplus \cdots \oplus a_n = 1$.

In sections 1 and 2 we develop some tools concerning the structure of Boolean metric spaces, while in sections 3 and 4 the main results are exposed. Namely, in section 3 we prove that if A is a CFG-ring and U is a subset of A^n, then U is an algebraic variety if and only if there are $x_1, \ldots, x_n$ in U such that any element of $x \in U$ is of the form $x = \sum_1^n a_i x_i$ where $a_1, \ldots, a_n \in B(A)$ and $a_1 \oplus \cdots \oplus a_n = 1$, if and only if there is distance-preserving bijection from U onto an algebraic variety $V \subset A^m$. Also, if $U \subset A^n$ and $V \subset A^m$ are algebraic varieties over A and $f : U \longrightarrow V$ is a map, the following are equivalent:

1. f is a polynomial map.
2. $d(f(x), f(y)) \leq d(x, y)$ for all x, y in U.
3. $f(\sum_1^n a_i x_i) = \sum_1^n a_i f(x_i)$ for all $x_1, \ldots, x_n$ in U and for all $a_1, \ldots, a_n$ in $B(A)$ with $a_1 \oplus \cdots \oplus a_n = 1$.

Thus, the category of algebraic varieties over an CFG-ring is equivalent to the category of those Boolean metric spaces over $B(A)$ that are isometric to some algebraic variety, that we have called CFG-spaces. Some special cases of these implications were known for $A = B$ a Boolean ring: that 1 is equivalent to 3 when $U = B^n$, $V = B$ is in Theorem 4.2 in [13], and that 1 is equivalent to 2 when $U = V = B$ was observed in [10].

In section 4, we present a classification of the Boolean metric spaces over a Boolean ring B, which is a classification of the algebraic varieties over a CFG-ring. We associate to each of those spaces a finite decreasing sequence of nonzero elements of B such that two spaces are isometric if and only if they have the same associated sequence.

The author wishes to thank professors Juan Martínez and Manuel Saorín, from University of Murcia, and Sergiu Rudeanu, from University of Bucharest, for their support and stimulus, and for their help in the redaction of this article.

1. Boolean metric spaces

A. Basic definitions and examples

Definition 1.1 *Let X be a set. A map $d : X \times X \longrightarrow B$ is said to be a Boolean metric if the following axioms hold, for all $x, y, z \in X$:*

1. $d(x, y) = 0$ *if and only if* $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) \vee d(y, z)$.

In that case, we will say that (X, d) is a metric space over B.

In the above definition, axiom 3 can be substituted by any of the following:

3'. $d(x, z)\overline{d(y, z)} \leq d(x, y)$
3". $d(x, z) + d(z, y) \leq d(x, y)$

Some suitable subsets of modules possess structure of Boolean metric space. We have called these subsets metrizable. Recall that the annihilator of an element x of a module over the ring A is the ideal $Ann(x) = \{a \in A : ax = 0\}$.

Definition 1.2 *Let A be a regular ring, M a module over A and X a subset of A. The set X will be said to be metrizable if for each $x, y \in X$ the ideal $Ann(x - y)$ is a principal ideal of A.*

If X is a metrizable subset of M, for each $x, y \in M$ the ideal $Ann(x-y)$ has a unique idempotent generator, say $a_{xy} \in B(A)$. Then, the map $d(x,y) = \overline{a_{xy}}$ is a Boolean metric on X, called the modular metric on X. Triangular inequality follows from

$$Ann(x-y) \cap Ann(y-z) \subseteq Ann(x-z).$$

For every $a \in A$ we have $Ann(a) = \overline{e(a)}A$, so A is a metrizable subset of itself and its modular metric is given by $d(x,y) = e(x-y)$. This is the same metric on A as defined in [8].

Furthermore, for every $n \in \mathbf{N}$, A^n is also a metrizable subset of itself and its modular metric is given by

$$d((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = e(x_1-y_1) \vee \cdots \vee e(x_n-y_n).$$

This is a particular case of the following general construction:

Definition 1.3 *Let $(X_1,d_1),\ldots,(X_n,d_n)$ be metric spaces over B. Then $(X_1 \times \cdots \times X_n, d)$ is also a metric space over B with*

$$d((x_1,\ldots,x_n),(y_1,\ldots,y_n)) := d_1(x_1,y_1) \vee \cdots \vee d_n(x_n,y_n)$$

This space will be called the product space of the spaces (X_i,d_i) and d will be called the product metric of the metrics d_i.

The formation of products is compatible with modular metrics:

Proposition 1.4 *Let S_i be a metrizable subset of the A-module M_i, for $i =, 1\ldots,n$. Then $S = S_1 \times \cdots \times S_n$ is a metrizable subset of $M_1 \times \cdots \times M_n$ and the modular metric in S equals the product metric of the modular metrics in the S_i's.*

PROOF: Call d_i the modular metric in S_i. For each $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ in S

$$\begin{aligned} Ann(x-y) &= \bigcap_{i=1}^{n} Ann(x_i-y_i) = \bigcap_{i=1}^{n} \overline{d_i(x,y)}A \\ &= \left(\prod_{i=1}^{n} \overline{d_i(x,y)}\right) A = \left(\overline{\bigvee_{i=1}^{n} d_i(x,y)}\right) A. \end{aligned}$$

□

Definition 1.5 *Let X and Y be Boolean metric spaces over B. A map $f : X \longrightarrow Y$ is said to be*

1. *contractive if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$.*
2. *an immersion if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.*
3. *an isometry if it is a bijective immersion.*

Contractive maps play the rôle of morphisms in the category of Boolean metric spaces over B, while isometries are the isomorphisms. Observe that immersions are always into.

Theorem 1.6 *Every metric space X over B is isometric to a metrizable subset of a B-module. Furthermore, if we fix $x_0 \in X$ there is a metrizable subset S of a B-module M such that $0 \in S$ and an isometry $g : X \longrightarrow S$ such that $g(x_0) = 0$.*

Proof: We define $f : X \longrightarrow B^X$ by $f(x) = (d(x, z))_{z \in X}$. To prove that $f(X)$ is metrizable and that $f : X \longrightarrow f(X)$ is an isometry, it is enough to see that $Ann(f(x) - f(y)) = \overline{d(x, y)}B$ for all $x, y \in X$. If $a \in Ann(f(x) + f(y))$ then,

$$a(d(x, z) + d(y, z))_{z \in X} = 0,$$

so for $z = x$, we have $ad(y, x) = 0$ and therefore $a \leq \overline{d(x, y)}$. Conversely, suppose $a \in \overline{d(x, y)}B$, then

$$a(d(x, z) + d(z, y)) \leq ad(x, y) = 0,$$

for all $z \in X$, so $a \in Ann(f(x) + f(y))$. For the last assertion, take $h : f(X) \longrightarrow f(X) + f(x_0)$ given by $h(x) = x + f(x_0)$. Then, h is an isometry between $f(X)$ and the metrizable set $S = f(X) + f(x_0)$ because $Ann(h(x) - h(y)) = Ann(x - y)$ for all x, y. Hence, the map $g = h \circ f$ is an isometry between X and S that verifies $g(x_0) = 0$. $\square$

B. Convex combinations and convex closures

Unless otherwise stated, X will be a metric space over B.

Definition 1.7 *Let $x_1, \dots, x_n \in X$ and let $a_1, \dots, a_n \in B$ such that $a_1 \oplus \cdots \oplus a_n = 1$. We will say that $x \in X$ is a convex combination of $x_1, \dots, x_n$ with coefficients $a_1, \dots, a_n$ if $a_i d(x, x_i) = 0$ for $i = 1, \dots, n$.*

Proposition 1.8 *If $x \in X$ is a convex combination of $x_1, \dots, x_n$ with coefficients $a_1, \dots, a_n$, then for all $y \in X$*

$$d(x, y) = \bigoplus_{i=1}^{n} a_i d(x_i, y)$$

Proof: For all $i = 1, \dots, n$, since $a_i d(x, x_i) = 0$, we have $a_i d(x_i, y) = a_i(d(x, x_i) + d(x_i, y)) \leq a_i d(x, y) \leq a_i(d(x, x_i) \vee d(x_i, y)) = a_i d(x_i, y)$, so $a_i d(x, y) = a_i d(x_i, y)$ and hence, we have $d(x, y) = (\sum_i a_i) d(x, y) = \sum_i a_i d(x_i, y)$. $\square$

Proposition 1.9 *If x and y are convex combinations of $x_1, \dots, x_n$ with coefficients $a_1, \dots, a_n$, then $x = y$.*

Proof: By Proposition 1.8

$$d(x, y) = \sum_{i=1}^{n} a_i d(x, x_i) = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j d(x_j x_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j d(x_j, x_i)$$

Note that if $i \neq j$ then $a_i a_j = 0$ and if $i = j$ then $d(x_j, x_i) = 0$, so all the terms in the above sum are zero, and therefore $d(x, y) = 0$. $\square$

Lemma 1.10 *Let S be a metrizable subset of an A-module M. Then,*

$$conv(S) = \{a_1 x_1 + \cdots + a_n x_n \in M : x_i \in S \ a_i \in B(A) \ \bigoplus_i a_i = 1\}$$

is also a metrizable subset of M.

Proof: Take $x, y \in conv(S)$, $x = \sum_1^n a_i x_i$ and $y = \sum_1^m b_j y_j$. Call $c_{ij} = a_i b_j$. Then, $\bigoplus_{i,j} c_{ij} = 1$ and $x = \sum_{i,j} c_{ij} x_i$ and $y = \sum_{i,j} c_{ij} y_j$. Hence, $Ann(x - y) = Ann(\sum_{i,j} c_{ij}(x_i - y_j)) = \sum_{i,j} c_{ij} Ann(x_i - y_j)$ which is a principal ideal because every $Ann(x_i - y_j)$ is principal (recall that, for regular rings, any finitely generated ideal is principal). $\square$

The following proposition will show that, when X is a metrizable subset of a module, convex combinations in (X, d) are exactly the corresponding linear combinations in the module.

Proposition 1.11 *Let S be a metrizable subset of an A-module, $x, x_1, \ldots, x_n \in S$ and $a_1, \ldots, a_n \in B$ such that $\bigoplus_{i=1}^n a_i = 1$. Then, x is a convex combination of $x_1, \ldots, x_n$ with coefficients $a_1, \ldots, a_n$ if and only if $x = a_1x_1 + \cdots + a_nx_n$.*

Proof: Suppose $x = a_1x_1 + \cdots + a_nx_n$. We must check that, for each $i \in \{1, \ldots, n\}$, $a_i d(x, x_i) = 0$. It is clear that $a_i \in Ann(x - x_i) = \overline{d(x, x_i)}A$, so $a_i \leq \overline{d(x, x_i)}$ and hence, $a_i d(x, x_i) = 0$.

Conversely, suppose $x \in S$ is a convex combination of $x_1, \ldots, x_n$ with coefficients $a_1, \ldots, a_n$. Let $y = \sum_1^n a_i x_i \in conv(S)$, which is metrizable, by Lemma 1.10. The implication that we have already proved, tells us that y is a convex combination of $x_1, \ldots, x_n$ with coefficients $a_1, \ldots, a_n$ in $conv(S)$. The same holds for x, so by Proposition 1.9, $x = y$. □

In general, in any metric space X, we will denote by $\sum_{i=1}^n a_i x_i$ or by $a_1x_1 + \cdots + a_nx_n$ the convex combination of $x_1, \ldots, x_n$ with coefficients $a_1, \ldots, a_n$, if it exists.

Recall that Theorem 1.6 allows us to identify any metric space X over B with a metrizable subset of a B-module, and then, by Theorem 1.11, convex combinations are just the corresponding linear combinations in the module and the metric is the modular metric.

Contractive maps can be characterized as those that preserve convex combinations.

Theorem 1.12 *For a map $f : X \longrightarrow Y$ between two metric spaces (X, d) and (Y, d') the following are equivalent:*

1. *f is contractive.*
2. *For all $x, x_1, \ldots, x_n \in X$ and $a_1, \ldots, a_n \in B$ with $\bigoplus_1^n a_i = 1$, if $x = \sum a_i x_i$, then $f(x) = \sum a_i f(x_i)$.*

PROOF: ($1 \Rightarrow 2$) Let $x = \sum_i a_i x_i$. Then, for every i, we have $0 = a_i d(x, x_i) \geq a_i d(f(x), f(x_i))$, so $f(x) = \sum a_i f(x_i)$.

($2 \Rightarrow 1$) Given $x, y \in X$, $x = d(x,y)x + \overline{d(x,y)}y$. Hence, by our assumption $f(x) = d(x,y)f(x) + \overline{d(x,y)}f(y)$ and making use of Proposition 1.8 we have finally

$$\begin{aligned} d(f(x), f(y)) &= d(x,y)d(f(x), f(y)) + \overline{d(x,y)}d(f(y), f(y)) \\ &= d(x,y)d(f(x), f(y)) \end{aligned}$$

and therefore $d(f(x), f(y)) \leq d(x,y)$. □

Given $x_1, \ldots, x_n \in X$ and $a_1, \ldots, a_n \in B$ with $\bigoplus a_i = 1$, there may exist no convex combination of the x_i's with coefficients a_i's. So we have the next definition:

Definition 1.13 *A metric space (X, d) over B is said to be convex if given any $x_1, \ldots, x_n \in X$ and any $a_1, \ldots, a_n \in B$ with $\bigoplus a_i = 1$, there exists in X the convex combination of the x_i's with coefficients the a_i's.*

This notion of convexity is different from the defined in [3].

Definition 1.14 *A convex closure of a metric space X is a convex metric space $Y \supseteq X$ such that any element in Y is a convex combination of elements of X.*

Observe that every metric space X over B has a convex closure, because X is isometric to a metrizable subset S of B-module and in this case, the set $conv(S)$ of Lemma 1.10 is a convex closure of S.

Theorem 1.15 *Let $X \subseteq \bar{X}$ and $Y \subseteq \bar{Y}$ be convex closures. Each contractive map $f : X \longrightarrow Y$ extends to a unique contractive map $\bar{f} : \bar{X} \longrightarrow \bar{Y}$. Furthermore,*

1. *$\bar{f}$ is immersion if and only if f is, and if f is isometry, so is $\bar{f}$.*
2. *For two contractive maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ we have $\overline{gf} = \bar{g}\bar{f}$.*

PROOF: For each element $x \in \bar{X}$, choose an expression of x as a convex combination of elements of X, like $x = \sum_i a_i x_i$. If we want $\bar{f}$

to be contractive it must be defined like $\bar{f}(x) = \sum_i a_i f(x_i) \in \bar{Y}$. This proves uniqueness. For existence we must check that, so defined, $\bar{f}$ is contractive. We take $x, y \in \bar{X}$, and their corresponding expressions $x = \sum a_i x_i$ and $y = \sum b_j y_j$ with $x_i, y_j \in X$:

$$d(\bar{f}(x), \bar{f}(y)) = \bigoplus a_i b_j d(f(x_i), f(y_j)) \leq \bigoplus a_i b_j d(x_i, y_j) = d(x, y)$$

If f is immersion then the inequality turns into an equality, and we deduce that $\bar{f}$ is an immersion. Property (2) is trivial and from this, using f^{-1}, we deduce that if f is isometry so is $\bar{f}$. $\square$

As a corollary, we get that the convex closure of a metric space is unique, up to isometry, since if $X \subseteq X_1, X_2$ are two convex closures of X, then 1_X extend to an isometry $f : X_1 \longrightarrow X_2$.

In the sequel $conv(X)$ will denote a convex closure of X. We finish by stating some elementary properties of convex spaces and convex closures.

Let X and Y be convex metric spaces over B and $U \subseteq X$. Then, the following hold:

1. The set of all convex combinations of elements of U in X is a convex closure of U(In this situation, the notation $conv(U)$ will refer to this set).
2. If $f : X \longrightarrow Y$ is contractive, then $f(conv(U)) = conv(f(U))$.
3. If $X_1, \ldots, X_n$ are metric spaces over B, then $conv(X_1) \times \cdots \times conv(X_n)$ is a convex closure of $X_1 \times \cdots \times X_n$.

2. CFG-spaces

Definition 2.1 *A metric space X over B will be said to be a CFG-space (convex finitely generated space) if it is the convex closure of a finite subspace.*

Observe that

1. If X is a CFG-space and $f : X \longrightarrow Y$ is contractive, then $f(X)$ is a CFG-space.

2. The product of a finite number of CFG-spaces is a CFG-space.

For technical reasons, it is convenient to work with pointed metric spaces. $(X, 0)$ is said to be a (pointed) metric space if X is a metric space over B and $0 \in X$. A map $f : (X, 0) \longrightarrow (Y, 0')$ will mean a map $f : X \longrightarrow Y$ such that $f(0) = 0'$, and expressions like $x \in (X, 0)$ will mean simply $x \in X$.

Throughout this section, we fix a convex metric space $(X, 0)$. By Theorem 1.6, it is not restrictive to suppose that X is a metrizable convex subset of a B-module M and that 0 is the zero element of M. In $(X, 0)$ we will use the following notations:

- For $x \in X$, $|x| := d(0, x)$.
- If $x_1, \dots, x_n \in X$ and $a_1, \dots, a_n \in B$ are such that $a_i a_j = 0$ whenever $i \neq j$, then we have an element of X:

  $$a_1x_1 + \cdots + a_nx_n = a_00 + a_1x_1 + \cdots + a_nx_n$$

  where $a_0 = 1 + a_1 + \cdots a_n$ (note that the right expression represents an element of X since $a_0 \oplus \cdots \oplus a_n = 1$ and X is a convex space). Such a combination will be called an orthogonal combination. As a particular case, $ax = ax + \bar{a}0$ for $x \in X$, $a \in B$.
- For $x \in X$, $Bx := \{ax : a \in B\} = conv(0, x)$.
- For $x, y \in X$, $x \star y := \overline{d(x, y)}x$.

Note that any contractive map $f : (X, 0) \longrightarrow (Y, 0')$ preserves orthogonal combinations. In the following lemma, we state some elementary properties:

Lemma 2.2 *Let $x, y \in X$ and $a, b \in B$. Then:*

1. *The maps $||: (X, 0) \longrightarrow (B, 0)$ and $x \star \_ : (X, 0) \longrightarrow (X, 0)$ are contractive, so both preserve orthogonal combinations.*
2. *$ax = bx$ if and only if $a + b \in \overline{|x|}B$ (if and only if $a + \overline{|x|}B = b + \overline{|x|}B$).*
3. *$ax = 0$ if and only if $a \leq \overline{|x|}$, and $ax = x$ if and only if $a \geq |x|$.*
4. *The operation $(\star)$ is commutative.*

Proof: For property 1, the function $x \star \_$ can be expressed as a composition of $y \mapsto d(x, y)$, $b \mapsto \bar{b}$ and $b \mapsto bx$ and all of them are contractive.

Property 2: Suppose X is a metrizable subset of a B-module. Then, $ax = bx$ if and only if $a + b \in Ann(x - 0) = \overline{d(x, 0)}B$.

Property 3 follows from 2.

For property 4, $x \star y = y \star x$ if and only if $\overline{d(x, y)}x + d(x, y)0 = \overline{d(x, y)}y + d(x, y)0$. This equality is easily checked verifying that the distance between the two terms is zero using Proposition 1.8. □

Lemma 2.3 *$Bx \cap By = B(x \star y)$ for all $x, y \in X$.*

Proof: Just by the definition of $\star$ we have $B(x \star y) \subseteq Bx$, and symmetrically, since $\star$ is commutative, $B(x \star y) \subseteq By$, so one inclusion is proved. Now suppose $u \in Bx \cap By$. Then $u = ax = by$, and if we call $c = ab$ then $cx = bax = bu = bby = u = aax = au = aby = cy$. Thus, $cx = u = cy$, and that implies $c \in Ann(x - y) = \overline{d(x, y)}B$ (suppose X is a subset of a module) and $u = cx = c\overline{d(x, y)}x = c(x \star y)$. □

Proposition 2.4 *For two elements $x, y \in X$ the following are equivalent:*

1. $x \star y = 0$
2. $Bx \cap By = \{0\}$
3. $d(x, y) = |x| \vee |y|$

In this case, x and y will be said to be orthogonal and we will write $x \perp y$.

Proof: $(1 \Leftrightarrow 2)$ is a direct consequence of Lemma 2.3. For $(1 \Leftrightarrow 3)$, we have (1) if and only if $0 = |x \star y| = \overline{d(x, y)}|x|$ and, by symmetry, if and only if $\overline{d(x, y)}|x| = 0 = \overline{d(x, y)}|y|$, which is equivalent to $|x|, |y| \leq d(x, y)$, and $|x| \vee |y| \leq d(x, y)$. The converse of the latter inequality is always true by axiom 3 of Boolean metric spaces. □

For $x, y \in (B, 0)$, we have $x \star y = \overline{d(x, y)}x = (x + y + 1)x = xy$, so this concept of orthogonality corresponds to *disjointness* in B.

Definition 2.5 *A finite subset $R \subseteq X$ will be said to be orthogonal if every two different elements in R are orthogonal, and $0 \notin R$. If, moreover, $X = conv(R \cup \{0\})$, R will be said to be a reference system or a referential of $(X, 0)$.*

Proposition 2.6 *Let $R = \{x_1, \ldots, x_n\}$ be a reference system of $(X, 0)$ and $x \in X$. There is a unique tuple $(a_1, \ldots, a_n) \in B^n$ satisfying the three following properties:*

1. *$a_i a_j = 0$ whenever $i \neq j$.*
2. *$\sum_1^n a_i x_i = x$.*
3. *$a_i \leq |x_i|$ for $i = 1, \ldots, n$.*

Such a tuple will be called the tuple of coordinates of x with respect to R.

Proof: Uniqueness: If $\sum_1^n a_i x_i = \sum_1^n b_i x_i$ in those conditions, multiplying by $a_i b_j$, $i \neq j$, we obtain $a_i b_j x_i = a_i b_j x_j \in Bx_i \cap Bx_j = \{0\}$ so for each i, $a_i x_i = a_i(\sum_j b_j) x_i = a_i b_i x_i$, and symmetrically $b_i x_i = a_i b_i x_i = a_i x_i$, so by Lemma 2.2 $a_i + b_i \in \overline{|x_i|} B$, and also $a_i + b_i \in |x_i| B$ since the a_i and b_i's are assumed to verify property 3. So $a_i + b_i = 0$ for all i.

Existence: Since $X = conv\{0, x_1 \ldots, x_n\}$, we can find $b_1, \ldots, b_n \in B$ verifying 1 and 2. Now set $a_i = |x_i| b_i$. The a_i's satisfy trivially 1 and 3. Using Lemma 2.2 we deduce from $a_i + b_i = \overline{|x_i|} b_i \in \overline{|x_i|} B$ that $a_i x_i = b_i x_i$ for all i. So $\sum_1^n a_i x_i = \sum_1^n b_i x_i = x$. $\square$

Proposition 2.7 *Let $R = \{x_1, \ldots, x_n\}$ be a referential of $(X, 0)$ and $(Y, 0')$ a convex metric space. Then, $f : R \longrightarrow Y$ is extensible to a (unique) contractive map $\hat{f} : (X, 0) \longrightarrow (Y, 0')$ if and only if $|f(x_i)| \leq |x_i|$ for $i = 1, \ldots, n$.*

Proof: Define f on $R \cup \{0\}$ by $f(0) = 0'$. By Theorem 1.15, f admits such an extension if and only if it is contractive. If f is contractive, it is clear that $|f(x_i)| \leq |x_i|$ for $i = 1, \ldots, n$, so one way is proved. Conversely, suppose $|f(x_i)| \leq |x_i|$ for every i. Then, for all $i \neq j$, since x_i and x_j are orthogonal, we have last equality in $d(f(x_i), f(x_j)) \leq |f(x_i)| \vee |f(x_j)| \leq |x_i| \vee |x_j| = d(x_i, x_j)$. $\square$

We check now that any CFG-space possesses a reference system.

Theorem 2.8 *Suppose* $X = conv\{0, x_1, \ldots, x_n\}$ *and that the set* $\{x_1, \ldots, x_s\}$ *is orthogonal. Then, there exist* $a_{s+1}, \ldots, a_n \in B$ *such that* $\{x_1, \ldots, x_s, a_{s+1}x_{s+1}, \ldots, a_n x_n\} \setminus \{0\}$ *is a referential of* $(X, 0)$.

Proof: Let $r = card\{(i,j) : x_i \star x_j \neq 0\}$. We make induction on r. We suppose that the theorem holds for any value lower than $r > 0$. We take x_i, x_j with $x_i \star x_j \neq 0$ and suppose, without loss of generality that $i, s < j$. Let $a := d(x_i, x_j)$. Since $ax_j \star x_i = a\overline{d(x_i, x_j)}x_i = 0$, we have $ax_j \perp x_i$. Also, $x_j = a(ax_j) + \bar{a}x_i$, and from this we deduce that $conv\{0, x_i, x_j\} = conv\{0, x_i, ax_j\}$ and therefore:

$$X = conv\{0, x_1, \ldots, x_{j-1}, ax_j, x_{j+1}, \ldots, x_n\}$$

Making use of the induction hypothesis, the proof is complete (in this system of generators there is at least one orthogonal pair more, since $x_i \perp ax_j$). $\square$

Corollary 2.9 *Let* $\{x_1, \ldots, x_s\}$ *be an orthogonal subset of the CFG-space* $(X, 0)$. *Then, there exist* $x_{s+1}, \ldots, x_n \in X$ *such that* $\{x_1, \ldots, x_n\}$ *is a referential of* $(X, 0)$.

In particular, any CFG-space $(X, 0)$ has a reference system.

Definition 2.10 *For* $U \subseteq X$, $U^\perp = \{x \in X : x \perp y \ \forall y \in U\}$.

Proposition 2.11 *For two CFG-spaces* $(U, 0) \subseteq (X, 0)$, *the space* $U^\perp$ *is a CFG-space and* $conv(U \cup U^\perp) = X$.

Proof: Let $\{x_1, \ldots, x_m\}$ be a reference system of $(U, 0)$ that we can extend to a reference system $\{x_1, \ldots, x_n\}$ of $(X, 0)$. We prove that $U^\perp = conv\{0, x_{m+1}, \ldots, x_n\}$. Take $x \in U^\perp$, $x = \sum_1^n a_i x_i$. Then, for $j = 1, \ldots, m$ we have $0 = x_j \star x = \sum_1^n a_i(x_i \star x_j) = a_j x_j$. Hence, $x = \sum_1^m a_i x_i$. $\square$

Proposition 2.12 *Let* $(U, 0) \subset (X, 0)$ *be two CFG-spaces,* $(Y, 0')$ *a convex metric space and* $f : (U, 0) \to (Y, 0')$ *and* $g : (U^\perp, 0) \to (Y, 0')$ *contractive maps. Then, there is a unique contractive map* $f \perp g : (X, 0) \to (Y, 0')$ *that extends* f *and* g.

Proof: We take a referential of $(U, 0)$ and another one of $(U^{\perp}, 0)$. The union is a referential of $(X, 0)$. Applying Theorem 2.7 the proposition is proved. □

3. Algebraic Geometry over CFG-rings

Definition 3.1 *A regular ring A is said to be a CFG-ring if, equipped with its modular metric, it is a CFG-space over $B(A)$.*

In this case, A^n (which is a metrizable A-module for which the product metric and the modular metric coincide) is a CFG-space over $B(A)$ too. If p is a prime number, any p-ring is a CFG-ring because if A is a p-ring, then $A = conv\{0, 1, \ldots, p-1\}$. A proof of this fact can be found in [15] (Corollary 1). There are CFG-rings that are not p-rings. For instance, take K a finite field and Ω a set. Then, K^{Ω} is regular and it is easy to see that the set of constant tuples constitute a finite system of generators of K^{Ω}, so K^{Ω} is a CFG-ring. The aim of this section is to prove Theorem 3.8.

Lemma 3.2 *Let R be a ring and $f : R^n \longrightarrow R$ a polynomial function. For every $x_1, \ldots, x_m \in R^n$ and every $e_1, \ldots, e_n \in B(R)$ such that $e_1 \oplus \cdots \oplus \cdots e_n = 1$, we have $f(\sum_i e_i x_i) = \sum_i e_i f(x_i)$.*

Proof: Let S be the set of all maps $g : R^n \longrightarrow R$ verifying the conclusion of the lemma. It is straightforward to check that the projections $\pi_i : R^n \longrightarrow R$ are in S (that proves the lemma for the polynomials $X_1, \ldots, X_n$), that constant maps are in S, and that the sums and products of maps in S lie in S. Any polynomial map is a sum of products of constants and the variables X_i's. □

Lemma 3.3 *Let A be a CFG-ring. A map $f : A^n \longrightarrow A^m$ is contractive if and only if it is a polynomial map.*

Proof: f is contractive if and only if all its components are, and the same holds about f being a polynomial map, so we can assume $m = 1$. The 'if' part is a consequence of Lemma 3.2, Theorem 1.12, and Proposition 1.11. For the 'only if' part, we assume that $f(0) = 0$

(it is sufficient to prove this case, just considering for an arbitrary f, the composition $h \circ f$ where $h : A \longrightarrow A$ is $h(x) = x + f(0)$). Take $\{x_1, \dots, x_r\}$ a referential of $(A^n, 0)$. We prove first the case $n = 1$:

Case $n = 1$. Consider the polynomial $g_i(x) = x \prod_{j \neq i}(x - x_j)$ for $i = 1, \dots, r$. Then,

$$
\begin{aligned}
e(g_i(x_i)) &= e(x_i) \prod_{j \neq i} e(x_i - x_j) = e(x_i) \prod_{j \neq i} d(x_i, x_j) \\
&= e(x_i) \prod_{j \neq i} (|x_i| \vee |x_j|) = |x_i| = e(x_i)
\end{aligned}
$$

Therefore, for each $i = 1, \dots, n$ there is a unit a_i of A such that $g_i(x_i) = a_i e(x_i)$. Consider the polynomial map $g : A \longrightarrow A$ given by

$$
g(x) = \sum_{i=1}^{r} a_i^{-1} f(x_i) g_i(x)
$$

Since $A = conv\{0, x_1, \dots, x_r\}$ and f and g are contractive, we prove that $g(x)$ and $f(x)$ coincide for $x = 0, x_1, \dots, x_r$. It is clear that $g(0) = 0 = f(0)$ because $g_i(0) = 0$. For $x = x_j$,

$$
g(x_j) = \sum_{i=1}^{r} a_i^{-1} f(x_i) g_i(x_j)
$$

and since $g_i(x_j) = 0$ whenever $i \neq j$,

$$
g_i(x_j) = a_j^{-1} f(x_j) g_j(x_j) = f(x_j) e(x_j)
$$

and this equals $f(x_j)$ because $|f(x_j)| \leq |x_j| = e(x_j)$.

General case: As a consequence of the case $n = 1$, we find that $e : A \longrightarrow A$ is a polynomial map, and therefore, for $v \in A^n$, the map $d(\sim, v) : A^n \longrightarrow A$ is polynomial too, since if $v = (a_1, \dots, a_n)$ then $d(x, v) = e(x_1 - a_1) \vee \dots \vee e(x_n - a_n)$ (recall that $x \vee y = x + y - xy$ for $x, y \in B(A)$). Hence, we can construct polynomial maps for $i = 1, \dots, r$ given by $G_i(x) = |x| \prod_{j \neq i} d(x, x_j)$. We define $G(x) = \sum_{i=1}^{r} f(x_i) G_i(x)$. and we are going to see that G and f coincide on $\{0, x_1, \dots, x_r\}$, so that, since both are contractive and this set generates A^n, that will prove that $f = G$, so f is a polynomial map. It is clear that $G(0) = 0 = f(0)$ because $G_i(0) = 0$ for all i. Since

$G_i(x_j) = 0$ if $i \neq j$,

$$G(x_i) = f(x_i)G_i(x_i) = f(x_i)|x_i| \prod_{i \neq j} d(x_i, x_j)$$
$$= f(x_i)|x_i| \prod_{j \neq i} |x_i| \vee |x_j| = f(x_i)|x_i| = f(x_i)$$

where the last equality follows from the fact that $|f(x_i)| \leq |x_i|$. $\square$

Lemma 3.4 *Let $(X, 0) = conv(H)$ be a convex metric space with $0 \in H$, and let $f : (X, 0) \to (Y, 0')$ be contractive. Then*

$$f^{-1}(0') = conv\{0, \overline{|f(x)|}x : x \in H\}$$

Proof: One inclusion is clear because all the elements that appear in the right term are in the convex set $f^{-1}(0')$. For the converse, if $x \in f^{-1}(0')$, in particular it is in X, so it can be expressed like $x = \sum_{i=1}^{n} a_i x_i$ with $x_i \in H$, and $a_i a_j = 0$ whenever $i \neq j$. Then,

$$0 = |f(x)| = a_1|f(x_1)| \oplus \cdots \oplus a_n|f(x_n)|,$$

and therefore $a_i|f(x_i)| = 0$, so $a_i = a_i\overline{|f(x_i)|}$ for all $i = 1, \ldots, n$. Finally,

$$x = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i \overline{|f(x_i)|} x_i \in conv\{0, \overline{|f(x_1)|}x_1, \ldots, \overline{|f(x_n)|}x_n\}.$$

$\square$

Note that a convex subset of a CFG-space need not be a CFG-space. For instance, $B = conv\{0, 1\}$ is a CFG-space, and those ideals of B that are not finitely generated are convex subsets that are not CFG-spaces.

Lemma 3.5 *Let X be a CFG-space. Then $Y \subseteq X$ is a CFG-space if and only if there exists a contractive map $f : X \to B$ such that $Y = f^{-1}(0)$.*

Proof: One way is a direct consequence of Lemma 3.4. For the converse, suppose Y is a CFG-space. If $Y = \emptyset$ it is trivial and if not, take $0' \in Y$ and $\{u_1, \ldots, u_k\}$ a referential in $(Y, 0')$ that we extend to a referential of $(X, 0')$, $\{u_1, \ldots, u_n\}$. By Theorem 2.7 we can define a contractive map $f : (X, 0') \to (B, 0)$ such that $f(u_i) = 0$ if $i \leq k$ and $f(u_i) = |u_i|$ if $i > k$. It is clear that $Y \subset f^{-1}(0)$ and for the other inclusion suppose $x \in f^{-1}(0)$ has coordinates $(a_1, \ldots, a_n)$. Then

$$0 = f(x) = f(\sum_{i=1}^{n} a_i u_i) = \bigoplus_{i=1}^{n} a_i f(u_i) = \bigoplus_{i=k+1}^{n} a_i |u_i| = \bigoplus_{i=k+1}^{n} a_i$$

so $a_i = 0$ for $i > k$, and $x = \sum_{i=1}^{k} a_i u_i \in conv\{0', u_1, \ldots, u_k\} = Y$. $\square$

Corollary 3.6 *If Y, Z are CFG-spaces contained in the space X, then $Y \cap Z$ is a CFG-space.*

Proof: If $Y = f^{-1}(0)$ and $Z = g^{-1}(0)$ with $f, g : conv(Y \cup Z) \to B$ contractive maps, then $Y \cap Z = (f \vee g)^{-1}(0)$. $\square$

Corollary 3.7 *Let $f : X \to Y$ be a contractive map between CFG-spaces. If $Z \subset Y$ is a CFG-space, then $f^{-1}(Z)$ is a CFG-space too.*

Proof: Let $g : Y \to B$ be such that $K = g^{-1}(0)$. Then $f^{-1}(K) = (g \circ f)^{-1}(0)$, so it is a CFG-space. $\square$

Theorem 3.8 *Let A be a CFG-ring.*

1. *A subset $U \subseteq A^n$ is an algebraic variety if and only if U is a CFG-metric subspace of A^n.*
2. *A map $f : U \longrightarrow V$ between two algebraic varieties is a polynomial map if and only if it is contractive.*

Proof: If U is an algebraic variety then, $U = \bigcap_1^k f_k^{-1}(0)$ where $f_i : A^n \longrightarrow A$ are polynomial maps, and therefore, by Lemma 3.3, contractive maps. Using Lemma 3.5 and Corollary 3.6 we deduce that U is a CFG-space. Conversely, If U is a CFG-space, by Lemma 3.5, there is a contractive map $f : A^n \longrightarrow B(A)$ with $U = f^{-1}(0)$. Then $U = g^{-1}(0)$ where g is the composition $A^n \longrightarrow B(A) \hookrightarrow A$, that is contractive and therefore polynomial, again by Lemma 3.3.

Suppose $f : U \longrightarrow V$ is contractive, choose some $u \in U$ and consider $f : (U, u) \longrightarrow (V, f(u))$ and $k : (U^{\perp}, u) \longrightarrow (V, f(u))$ the constant map. Since U and A^n are CFG-spaces and V is convex, we can consider, by Proposition 2.12, $f \perp k : A^n \longrightarrow A^m$ that is contractive, and therefore, a polynomial map, that extends f. The converse is a direct consequence of Lemma 3.3. □

4. Structure Theorem for CFG-spaces

We shall classify now CFG-spaces up to isomorphism. Reference systems do not give good isomorphism invariants, since they are not *unique up to isometry* (for instance, $\{1\}$ and $\{a, \bar{a}\}$ are non-isometric referentials of $(B, 0)$). The right concept for this purpose is the following:

Definition 4.1 *A referential $\{x_1, \ldots, x_n\}$ of $(X, 0)$ is said to be a base of $(X, 0)$ if $|x_1| \geq |x_2| \geq \cdots \geq |x_n|$.*

We will prove that there exists a base for any pointed CFG-space $(X, 0)$, and that they are unique in the sense of Theorem 4.6 below. We prove uniqueness first, and existence afterwards.

Definition 4.2 *Let $k > 0$ be an integer and X a metric space over B. The k-ideal of X (denoted by $I_k(X)$) is the ideal of B generated by*

$$\{ \prod_{0 \leq i < j \leq k} d(u_i, u_j) : u_0, \ldots, u_k \in X \}.$$

If $I_k(X)$ is principal, we will denote by $\alpha_k(X)$ its generator.

Lemma 4.3 *If $X = conv(H)$, then $I_k(H) = I_k(X)$ for all $k \in \mathbf{N}$.*

Proof: If U is any Boolean metric space, the map $f_U : U^{k+1} \to B$ given by $f_U(u_0, \ldots, u_k) = \prod_{0 \leq i < j \leq k} d(u_i, u_j)$ is contractive because it is a composition of *distance functions* and a polynomial function (the product). With this notation, $I_k(U)$ is the ideal generated by the image of f_U, and

$$\begin{aligned} Im(f_X) &= f_X(X^{k+1}) = f_X(conv(H^{k+1})) = conv(f_X(H^{k+1})) \\ &= conv(Im(f_H)) \end{aligned}$$

so both images generate the same ideal. □

Lemma 4.4 *Let X be a CFG-space. Then $I_k(X)$ is principal for all $k \in \mathbf{N}$ and there exists $n \in \mathbf{N}$ such that $I_k(X) = 0$ for all $k \geq n$. Hence, $\alpha_k(X)$ exists for all $k \in \mathbf{N}$ and $\alpha_k(X) = 0$ for $k \geq n$.*

Proof: Suppose $X = conv(H)$ with H finite. Then, $I_k(X) = I_k(H)$ is always a finitely generated ideal of B, so it is principal, and if we take $n = card(H)$, $0 = I_k(H) = I_k(X)$ if $k \geq n$. □

Lemma 4.5 *Let $\{x_1, \dots, x_n\}$ be a base of $(X, 0)$. Then, $\alpha_k(X) = |x_k|$ for $k \leq n$ and $\alpha_k(X) = 0$ if $k > n$.*

Proof: By Lemma 4.3, $\alpha_k(X) = \alpha_k(H)$ where $H = \{0, x_1, \dots, x_n\}$. If $k > n$, it is trivial that $\alpha_k(H) = 0$. If $k \leq n$, call y_i's to the reordering of the x_i's such that $0 = |y_0| \leq |y_1| \leq \cdots \leq |y_n|$ ($y_r = x_{n-r+1}$ if $r > 0$). For $i < j$ we have $d(y_i, y_j) = |y_i| \vee |y_j| = |y_j|$, by orthogonality. We wonder whether $I_k(H) = |x_k|B(= |y_{n-k+1}|B)$. One inclusion is because

$$|y_{n-k+1}| = \prod_{n \geq i > n-k} |y_i| = \prod_{n \geq i > j \geq n-k} d(y_i, y_j)$$

is one of the generators of $I_k(H)$. For the other inclusion we shall check that all the generators of $I_k(H)$ are in the ideal $|y_{n-k+1}|B$. Take $U = \{u_0, \dots, u_k\} \subseteq H$. By a cardinality argument, there must exist indices $r < s \leq n - k + 1$ such that $y_r, y_s \in U$, so

$$\prod_{0 \leq i < j \leq k} d(u_i, u_j) \leq d(y_r, y_s) = |y_s| \leq |y_{n-k+1}|$$

□

Theorem 4.6 *If* $\{x_1, \ldots, x_n\}$ *is a base of* $(X, 0)$ *and* $\{y_1, \ldots, y_m\}$ *is a base of* $(X, 0')$, *then* $n = m$ *and* $|x_i| = |y_i|$ *for* $i = 1, \ldots, n$. *Moreover, there exists an isometry* $f : (X, 0) \longrightarrow (X, 0')$ *such that* $f(x_i) = y_i$ *for* $i = 1, \ldots, n$.

PROOF: By Lemma 4.5, we know that $n = \max\{k : \alpha_k(X) \neq 0\} = m$ and $|x_i| = \alpha_i(X) = |y_i|$. About the last assertion, there exists a contractive map $f : (X, 0) \longrightarrow (X, 0')$ such that $f(x_i) = y_i$ for $i = 1, \ldots, n$, by virtue of Proposition 2.7. It is an isometry because we can find its inverse in an analogue way $g : (X, 0') \longrightarrow (X, 0)$ with $g(y_i) = x_i$. □

Lemma 4.7 *If* $(V, 0) \subset (X, 0)$ *are CFG-spaces and* $V^\perp = \{0\}$, *then* $V = X$.

PROOF: A reference system of $(V, 0)$, $\{x_1, \ldots, x_m\}$ can be extended to a referential of $(X, 0)$, $\{x_1, \ldots, x_n\}$. Then, $x_{m+1}, \ldots, x_n \in V^\perp = \{0\}$, so $V = conv\{0, x_1, \ldots, x_m\} = conv\{0, x_1, \ldots, x_n\} = X$. □

Lemma 4.8 *Let* X *be a CFG-space and* $f : X \longrightarrow B$ *contractive. Then, there exists* $u \in X$ *such that* $f(u) = \max\{f(x) : x \in X\}$.

PROOF: Suppose $X = conv\{x_0, \ldots, x_n\}$. The set $f(X) \subseteq B$ is closed under the operation $(\vee)$, because it is convex and $a \vee b = aa + \bar{a}b$. Hence, there exists $u \in X$ such that $f(u) = f(x_0) \vee \cdots \vee f(x_n)$. If $x \in X$, we express it as a convex combination $x = \sum_i a_i x_i$ and $f(x) = \bigoplus_i a_i f(x_i) = \bigvee_i a_i f(x_i) \leq \bigvee_i f(x_i) = f(u)$. □

Theorem 4.9 *Any CFG-space* $(X, 0)$ *possesses a base.*

PROOF: We define by recursion a sequence $(x_n)_{n=1}^\infty$ in X and a sequence $(U_n)_{n=1}^\infty$ of CFG-spaces contained in X:

- x_1 is such that $|x_1| = \max\{|x| : x \in X\}$; $U_1 = conv\{0, x_1\}$
- Given x_i and U_i for $i < n$, we take x_n such that $|x_n| = \max\{|x| : x \in U_{n-1}^\perp\}$ and $U_n := conv\{0, x_1, \ldots, x_n\}$

Note that those maximums exist by virtue of Lemma 4.8, since $U_{n-1}^\perp$ is a CFG-space by Proposition 2.11. The x_i's form an orthogonal set and verify $|x_i| \geq |x_j|$ whenever $i < j$. Therefore, $\{x_1, \ldots, x_n\} \setminus \{0\}$ is a base of $(U_n, 0)$. Since X is a CFG-space, by Lemma 4.4, there must exist some $k > 0$ with $0 = \alpha_k(X) \geq \alpha_k(U_k) = |x_k|$. So, taking r the largest integer such that $|x_r| \neq 0$, we have, just by the definition of $x_{r+1} = 0$, that $U_r^{\perp} = \{0\}$. Therefore, by Lemma 4.7, $U_r = X$ and we have already shown that $\{x_1, \ldots, x_r\} \setminus \{0\}$ is a base of $(U_r, 0)$. $\square$

Theorem 4.10 *Two CFG-spaces X and Y are isometric if and only if $\alpha_k(X) = \alpha_k(Y)$ for all $k \in \mathbf{N}$.*

PROOF: Suppose $\alpha_k(X) = \alpha_k(Y)$ for all k. Choose $0 \in X$, $0' \in Y$ and bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ of $(X, 0)$ and $(Y, 0')$ respectively. Then, $n = \max\{k : \alpha_k(X) = \alpha_k(Y) \neq 0\} = m$ and we can construct an isometry like in the proof of Theorem 4.6. $\square$

References

1. Batbedat, A. Distance Booléenne sur un 3-anneau. Enseignement Math. **1971**, *17*, 165–185.
2. Blumenthal, L. M. Boolean Geometry I. Rend. Circ. Mat. Palermo. **1952**, *1*, 343-360.
3. Blumenthal, L. M. Penning C.J. Boolean Geometry II. Rend. Circ. Mat. Palermo. **1961**, *10*, 175-192.
4. Ellis, D. Autometrized Boolean algebras I. Fundamental distance-theoretic properties of B. Canadian J. Math. **1951**, *3*, 87-93.
5. Ellis, D. Sprinkle, H.D. Topology of B-metric spaces. Compositio Math. **1956**, *12*, 250-262.
6. Goodearl, K. R. *Von Neumann Regular Rings*. Krieger. Malabar, Florida, 1979.
7. Hararay, F. Melter, R.A. Peled, U.N. Tomescu, I. Boolean distance for graphs. Discrete Math. **1982**, *39*, 123-127.
8. Melter, R. A. Boolean valued rings and Boolean metric spaces. Arch. Math. **1964**, *15*, 354-363.

9. Melter, R. A. Contributions to Boolean geometry of p-rings. Pac. J. Math. **1964**, *14*, 995-1017.
10. Melter, R. A. Rudeanu, S. Characterizations of Boolean functions. An. Stiint. Univ. Al. I. Cuza Iasi. **1982**, *18*, 161-169.
11. Melter, R. A. Rudeanu, S. Geometry of 3-rings. Colloq. Math. Soc. Janos Bolyai. **1974**, *14*, 249-269.
12. Popescu, N. Vraciu, C. Sur la structure des anneaux absolument plats commutatifs. Journal of Algebra. **1976**, *40*, 364-383.
13. Rudeanu, S. *Boolean Functions and Equations.* North-Holland, Amsterdam, 1974.
14. Rudeanu, S. *Lattice Functions and Equations.* Springer, London, 2001.
15. Zemmer, J.L. Some remarks on p-rings and their Boolean geometry. Pac. J. Math. **1956**, *6*, 193-208.