MEASURABLE SELECTORS AND SET-VALUED PETTIS INTEGRAL IN NON-SEPARABLE BANACH SPACES

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Abstract. Kuratowski and Ryll-Nardzewski’s theorem about the existence of measurable selectors for multi-functions is one of the keystones for the study of set-valued integration; one of the drawbacks of this result is that separability is always required for the range space. In this paper we study Pettis integrability for multi-functions and we obtain a Kuratowski and Ryll-Nardzewski’s type selection theorem without the requirement of separability for the range space. Being more precise, we show that any Pettis integrable multi-function \( F : \Omega \to \text{cwk}(X) \) defined in a complete finite measure space \((\Omega, \Sigma, \mu)\) with values in the family \( \text{cwk}(X) \) of all non-empty convex weakly compact subsets of a general (non-necessarily separable) Banach space \( X \) always admits Pettis integrable selectors and that, moreover, for each \( A \in \Sigma \) the Pettis integral \( \int_A F d\mu \) coincides with the closure of the set of integrals over \( A \) of all Pettis integrable selectors of \( F \). As a consequence we prove that if \( X \) is reflexive then every scalarly measurable multi-function \( F : \Omega \to \text{cwk}(X) \) admits scalarly measurable selectors; the latter is also proved when \((X^*, w^*)\) is angelic and has density character at most \( \omega_1 \). In each of these two situations the Pettis integrability of a multi-function \( F : \Omega \to \text{cwk}(X) \) is equivalent to the uniform integrability of the family \( \{ \sup x^*(F(\cdot)) : x^* \in B_{X^*} \} \subset \mathbb{R}^\Omega \). Results about norm-Borel measurable selectors for multi-functions satisfying stronger measurability properties but without the classical requirement of the range Banach space being separable are also obtained.

1. INTRODUCTION

Set-valued integration has its origin in the seminal papers by Aumann [2] and Debreu [9] and has been a very useful tool in areas like optimization and mathematical economics. The set-valued Pettis integral theory, which goes back to the monograph by Castaing and Valadier [7], has attracted recently the attention of several authors, see for instance [1, 5, 6, 11, 12, 15, 24, 44] and [45]. All these studies deal with multi-functions whose values are subsets of a Banach space \( X \) that is always assumed to be separable. The main reason for this limitation on \( X \) relies on the fact that an integrable multifunction should have integrable (measurable) selectors and the tool to find these measurable selectors has always been the well-known selection theorem of Kuratowski and Ryll-Nardzewski [29] that only works when the range space is separable. For a detailed account on measurable selection results and set-valued integration we refer the reader to the monographs [7, 27] and the survey [23].

Our main goal here is to show that most of the Pettis integral theory for multi-functions can be done without the restriction of separability on the range space. The extension from the separable case to the non-separable one is not so obvious and to do so we have to obtain a number of new measurable selection results for multi-functions in the non-separable case.

Throughout this paper \((\Omega, \Sigma, \mu)\) is a complete finite measure space, \( X \) is a Banach space and \( \text{cwk}(X) \) (resp. \( \text{ck}(X) \)) denotes the family of all convex weakly compact (resp. norm compact) non-empty subsets of \( X \). We write \( \delta^*(x^*, C) := \sup \{ x^*(x) : x \in C \} \) for any
bounded set $C \subset X$ and $x^* \in X^*$. A multi-function $F : \Omega \to cwk(X)$ is said to be Pettis integrable if

- $\delta^*(x^*, F)$ is integrable for each $x^* \in X^*$;
- for each $A \in \Sigma$, there is $\int_A F \, d\mu \in cwk(X)$ such that
  $$\delta^*(x^*, \int_A F \, d\mu) = \int_A \delta^*(x^*, F) \, d\mu \quad \text{for every } x^* \in X^*.$$

Here the function $\delta^*(x^*, F) : \Omega \to \mathbb{R}$ is defined by $\delta^*(x^*, F)(\omega) = \delta^*(x^*, F(\omega))$.

The paper is organized as follows. In Section 2 we study Pettis integrable multi-functions via their selectors. Our Theorem 2.5 states that every Pettis integrable multifunction $F : \Omega \to cwk(X)$ admits indeed Pettis integrable selectors. Moreover, in this case, for each $A \in \Sigma$ the integral $\int_A F \, d\mu$ coincides with the closure of the set of integrals over $A$ of all Pettis integrable selectors of $F$, Theorem 2.6. In the previous statement, the “closure” can be dropped provided that $X^*$ is $w^*$-separable, Corollary 2.7. These results are the non-trivial extension of part of Theorem A below that is considered as the milestone result in the set-valued Pettis integral theory for separable Banach spaces.

**Theorem A** ([8, 44, 45] and [7, Chapter V, §4]). Let $X$ be a separable Banach space and $F : \Omega \to cwk(X)$ a multi-function. The following conditions are equivalent:

(i) $F$ is Pettis integrable.
(ii) The family $W_F = \{\delta^*(x^*, F) : x^* \in B_{X^*}\}$ is uniformly integrable.
(iii) The family $W_F$ is made up of measurable functions and any scalarly measurable selector of $F$ is Pettis integrable.

In this case, for each $A \in \Sigma$ the integral $\int_A F \, d\mu$ coincides with the set of integrals over $A$ of all Pettis integrable selectors of $F$.

To get ready for the proof of a full counterpart to Theorem A for non-separable Banach spaces we quote in Section 3 some known facts about the existence of countably additive selectors and the Orlicz-Pettis theorem for multi-measures which are due to Godet-Thobie [20], Costé [8] and Pallu de la Barrière [33]: new proofs for these results are included.

In Section 4 we discuss the possible extensions of Theorem A to the non-separable setting. The implications (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) hold without any assumption on $X$, Theorem 4.1 and Corollary 2.3. We show in Theorem 4.2 that the equivalence (i)$\iff$(iii) holds true if $X$ has the following property: every scalarly measurable multi-function $F : \Omega \to cwk(X)$ (meaning that $\delta^*(x^*, F)$ is measurable for all $x^* \in X^*$) admits a scalarly measurable selector. This condition, which we call Scalarly Measurable Selector Property with respect to $\mu$, shortly $\mu$-SMSP, is shared by many Banach spaces besides the separable ones, as explained a few lines below. On the other hand, to prove (ii)$\Rightarrow$(i) we have to require, in addition to the $\mu$-SMSP, that $X$ has the so-called Pettis Integral Property with respect to $\mu$ (shortly $\mu$-PIP), Corollary 4.3. The last part of Section 4 is devoted to characterize Pettis integrability of multi-valued functions via single-valued ones and we pay particular attention to the the case of multi-functions with norm compact values.

In Section 5 we are concerned with the existence of “measurable” selectors for multi-functions $F : \Omega \to cwk(X)$ which satisfy one of the following measurability properties:

(a) $\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} \in \Sigma$ for every closed half-space $M \subset X$ (equivalently, $F$ is scalarly measurable).
(b) $\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} \in \Sigma$ for every convex closed set $M \subset X$.
(c) $\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} \in \Sigma$ for every norm closed set $M \subset X$.

When $X$ is separable, it is known that (a), (b) and (c) are equivalent to the Effros measurability of $F$ (i.e. the same property than (γ) but replacing “closed” by “open”, cf. [7, Theorem III.37]). In this case, the selection theorem of Kuratowski and Ryll-Nardzewski, cf. [7, Theorem III.30], ensures that such an $F$ admits a $\text{Borel}(X, \text{norm})$-measurable (hence
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strongly measurable) selector. In the non-separable case these measurability notions are not equivalent in general and the situation becomes more complicated. Subsection 5.1 is started with Theorem 5.1 by proving that reflexive Banach spaces have \( \mu \)-SMSP. Beyond that, our Theorem 5.4 shows that many other Banach spaces have \( \mu \)-SMSP: for instance, this happens if the dual space is \( w^* \)-angelic and has \( w^* \)-density character less than or equal to the uncountable cardinal number \( \kappa (\mu) \), Example 5.5 – we recall that the class of Banach spaces having \( w^* \)-angelic dual is very large and contains all weakly Lindelöf determined spaces and, in particular, all weakly compactly generated ones. Amongst other things we provide in Theorem 5.15 a different proof of Valadier’s result [43] saying that spaces with \( w^* \)-separable dual also have \( \mu \)-SMSP. To this end we prove that a multi-function \( F : \Omega \to cwk(X) \) is scalarly measurable if and only if \( \{ \omega \in \Omega : F(\omega) \cap M \neq \emptyset \} \in \Sigma \) for every set \( M \subset X \) which can be written as a finite intersection of closed half-spaces, Theorem 5.10. The paper is closed by Subsection 5.2 where we study the existence of Borel(\( X, \) norm)-measurable selectors for multi-functions \( F : \Omega \to cwk(X) \) satisfying (\( \beta \)). We prove, for instance, that such selectors always exist provided that \( X \) admits an equivalent locally uniformly rotund norm, Corollary 5.19: this improves a result by Leese [30] who obtained the same conclusion for multi-functions satisfying (\( \gamma \)) when \( X \) admits an equivalent uniformly rotund norm.

Terminology. Our unexplained terminology can be found in our standard references for multi-functions [7, 27], Banach spaces [16] and vector integration [13, 42].

The cardinality of a set \( \Gamma \) is denoted by card(\( \Gamma \)). The cardinality of \( \mathbb{N} \) (resp. \( \mathbb{R} \)) is denoted by \( \aleph_0 \) (resp. \( \epsilon \)). The symbol \( \omega_1 \) stands for the first uncountable ordinal.

Our topological spaces \( (T, \Xi) \) are always assumed to be Hausdorff. The density character of \( (T, \Xi) \), denoted by dens(\( T, \Xi \)) or simply by dens(\( T \)), is the minimal cardinality of a dense set in \( T \).

All vector spaces here are assumed to be real. Given a subset \( S \) of a vector space, we write \( co(S) \) and \( span(S) \) to denote, respectively, the convex and linear hull of \( S \). By letters \( X \) and \( Y \) we always denote Banach spaces. \( B_Y \) is the closed unit ball of \( Y \) and \( Y^* \) stands for the topological dual of \( Y \). Given \( y^* \in Y^* \) and \( y \in Y \), we write either \( \langle y^*, y \rangle \) or \( y^*(y) \) to denote the evaluation of \( y^* \) at \( y \). The weak (resp. weak*) topology on \( Y \) (resp. \( Y^* \)) is denoted by \( w \) (resp. \( w^* \)). Given a non-empty set \( \Gamma \) (resp. a compact topological space \( K \)), we write \( \ell_\infty(\Gamma) \) (resp. \( C(K) \)) to denote the Banach space of all bounded (resp. continuous) real-valued functions on \( \Gamma \) (resp. \( K \)), equipped with the supremum norm.

A function \( f : \Omega \to Y \) is said to be scalarly measurable if, for each \( y^* \in Y^* \), the composition \( \langle y^*, f \rangle := y^* \circ f : \Omega \to \mathbb{R} \) is measurable. By a result of Edgar [14], \( f \) is scalarly measurable if and only if it is Baire\( (Y, w) \)-measurable. Recall also that \( f \) is said to be Pettis integrable if

(i) \( y^* \circ f \) is integrable for every \( y^* \in Y^* \);

(ii) for each \( A \in \Sigma \), there is an element \( \int_A f \, d\mu \in Y \) such that

\[
\langle y^*, \int_A f \, d\mu \rangle = \int_A y^* \circ f \, d\mu \quad \text{for every } y^* \in Y^*.
\]

A function \( f : \Omega \to Y \) is strongly measurable if it is the \( \mu \)-a.e. limit of a sequence of simple functions or, equivalently, if it is Borel\( (Y, \text{norm}) \)-measurable (or just scalarly measurable) and there is \( E \in \Sigma \) with \( \mu(\Omega \setminus E) = 0 \) such that \( f(E) \) is separable, cf. [13, Theorem 2, p. 42].

2. Set-valued Pettis integral and selectors

In order to prove our main result in this section stating that any Pettis integrable multifunction admits Pettis integrable selectors, Theorem 2.5, we need some previous work.

Recall first that a function \( \varphi : X^* \to \mathbb{R} \) is said to be positively homogeneous if \( \varphi(\alpha x^*) = \alpha \varphi(x^*) \) for every \( \alpha > 0 \) and \( x^* \in X^* \). \( \varphi \) is said to be subadditive if
\[ \varphi(x^* + y^*) \leq \varphi(x^*) + \varphi(y^*) \] for all pairs \((x^*, y^*) \in X^* \times X^*\). \(\varphi\) is said to be sublinear if it is both positively homogeneous and subadditive. We note that if \(C \in \text{cwk}(X)\) then the map \(x^* \mapsto \delta^*(x^*, C)\) is a sublinear functional in \(X^*\) that is \(\tau(X^*, X)\)-continuous. Here \(\tau(X^*, X)\) stands for the Mackey topology on \(X^*\), that is, the topology of uniform convergence on weakly compact subsets of \(X\), cf. [28, §21.4]. Recall that, by the Mackey-Arens theorem, \(\tau(X^*, X)\) is the finest locally convex topology on \(X^*\) whose topological dual is \(X\), hence the \(\text{w}^*\)-closure and the \(\tau(X^*, X)\)-closure of any convex set \(C \subset X^*\) coincide, cf. [28, §21.4(2) and §20.8(6)].

**Lemma 2.1.** Let \(F : \Omega \to \text{cwk}(X)\) be a multi-function such that \(\delta^*(x^*, F)\) is integrable for every \(x^* \in X^*\). The following statements are equivalent:

(i) \(F\) is Pettis integrable.

(ii) For each \(A \in \Sigma\), the mapping \(\varphi^F_A : X^* \to \mathbb{R}, \ x^* \mapsto \int_A \delta^*(x^*, F) \ d\mu\), is \(\tau(X^*, X)\)-continuous.

**Proof.** The implication (i)⇒(ii) follows from the fact that

\[ \delta^*(x^*, \int_A F \ d\mu) = \int_A \delta^*(x^*, F) \ d\mu \quad \text{for every } x^* \in X^*, \]

and the \(\tau(X^*, X)\)-continuity of the map \(x^* \mapsto \delta^*(x^*, \int_A F \ d\mu)\). Conversely, assume that (ii) holds and fix \(A \in \Sigma\). Since \(\varphi^F_A\) is a sublinear function, it is convex. This fact and the \(\tau(X^*, X)\)-continuity of \(\varphi^F_A\) allow us to deduce that for every \(t \in \mathbb{R}\) the set \(\{x^* \in X^* : \varphi^F_A(x^*) \leq t\}\) is convex and \(\tau(X^*, X)\)-closed, hence \(\text{w}^*\)-closed. Therefore \(\varphi^F_A\) is \(\text{w}^*\)-lower semicontinuous and [7, Theorem II-16] applies to provide us with a non-empty convex, closed and bounded set \(C \subset X\) such that \(\varphi^F_A(x^*) = \delta^*(x^*, C)\) for every \(x^* \in X^*\). Finally, the fact that \(\varphi^F_A\) is \(\tau(X^*, X)\)-continuous can be applied again to conclude that \(C\) is weakly compact. Indeed, the set \(U := \{x^* \in X^* : \varphi^F_A(x^*) < 1\} \cap \{x^* \in X^* : \varphi^F_A(-x^*) < 1\}\) is a \(\tau(X^*, X)\)-neighborhood of 0 and thus its polar \(U^\circ = \{x \in X : |x(x^*)| \leq 1\}\) for all \(x^* \in U\) is weakly compact, [28, §21.4.1]. Since \(C\) is weakly closed and contained in \(U^\circ\), \(C\) is weakly compact as well.

Observe that for every bounded set \(C \subset X\) and every \(x^* \in X^*\) we have

\[ \inf \{x^*(x) : x \in C\} = -\delta^*(-x^*, C). \]

**Lemma 2.2.** Let \(F, G : \Omega \to \text{cwk}(X)\) be two multi-functions such that \(F\) is Pettis integrable, \(G\) is scalarly measurable and, for each \(x^* \in X^*\), we have \(\delta^*(x^*, G) \leq \delta^*(x^*, F)\) \(\mu\text{-a.e.}\). Then \(G\) is Pettis integrable and \(\int_A G \ d\mu \subset \int_A F \ d\mu\) for every \(A \in \Sigma\).

**Proof.** Given \(x^* \in X^*\), we have \(-\delta^*(-x^*, F) \leq \delta^*(x^*, G) \leq \delta^*(x^*, F)\) \(\mu\text{-a.e.}\). and so \(\delta^*(x^*, G)\) is integrable. Fix \(A \in \Sigma\). The mapping \(\varphi^G_A\) is subadditive and satisfies

\[ \varphi^G_A(x^*) \leq \varphi^F_A(x^*) \]

for all \(x^* \in X^*\), hence

\[ |\varphi^G_A(x^*) - \varphi^G_A(y^*)| \leq |\varphi^F_A(x^* - y^*)| + |\varphi^F_A(y^* - x^*)| \]

for every \(x^*, y^* \in X^*\). Since \(F\) is Pettis integrable, \(\varphi^F_A\) is \(\tau(X^*, X)\)-continuous and the previous inequality implies that \(\varphi^G_A\) is also \(\tau(X^*, X)\)-continuous. Since \(A \in \Sigma\) is arbitrary, an appeal to Lemma 2.1 ensures that \(G\) is Pettis integrable. Moreover, for each \(A \in \Sigma\) we have \(\int_A G \ d\mu \subset \int_A F \ d\mu\), by the Hahn-Banach separation theorem and the fact that

\[ \delta^*(x^*, \int_A G \ d\mu) = \int_A \delta^*(x^*, G) \ d\mu \leq \int_A \delta^*(x^*, F) \ d\mu = \delta^*(x^*, \int_A F \ d\mu) \]

for every \(x^* \in X^*\). The proof is over.
Given a multi-function $F : \Omega \to cwk(X)$ and $A \in \Sigma$ we write
\[ IS_F(A) := \left\{ \int_A f \, d\mu : f \text{ is a Pettis integrable selector of } F \right\}. \]

Note that $IS_F(A)$ might be empty in general and that otherwise it is a convex subset of $X$.

Next corollary says, in particular, that $IS_F(A) \subset \int_A F \, d\mu$ whenever $F$ is Pettis integrable.

**Corollary 2.3.** Let $F : \Omega \to cwk(X)$ be a Pettis integrable multi-function. If $f : \Omega \to X$ is a scalarly measurable selector of $F$, then $f$ is Pettis integrable and
\[ \int_A f \, d\mu \in \int_A F \, d\mu \quad \text{for every } A \in \Sigma. \]

**Proof.** Apply Lemma 2.2 to the multi-function $G(\omega) := \{ f(\omega) \}$.

To prove the main result of this section we also need the following lemma:

**Lemma 2.4** ([43, Lemme 3]). Let $F : \Omega \to cwk(X)$ be a scalarly measurable multi-function. Fix $x_0^* \in X^*$ and consider the multi-function
\[ G : \Omega \to cwk(X), \quad G(\omega) := \{ x \in F(\omega) : x_0^*(x) = \delta^*(x_0^*, F(\omega)) \}. \]

Then $G$ is scalarly measurable.

**Theorem 2.5.** Let $F : \Omega \to cwk(X)$ be a Pettis integrable multi-function. Then $F$ admits a Pettis integrable selector.

**Proof.** Since $\int_A F \, d\mu \in cwk(X)$, we can find an exposed point $x_0 \in \int_A F \, d\mu$ (cf. [4, Theorem 3.6.1]), that is, there is some $x_0^* \in X^*$ such that $x_0^*(x_0) > x_0^*(x)$ for every $x \in \int_A F \, d\mu \setminus \{x_0\}$. Let us consider the multi-function
\[ G : \Omega \to cwk(X), \quad G(\omega) := \{ x \in F(\omega) : x_0^*(x) = \delta^*(x_0^*, F(\omega)) \}. \]

By Lemma 2.4, $G$ is scalarly measurable. Since $G(\omega) \subset F(\omega)$ for every $\omega \in \Omega$ and $F$ is Pettis integrable, an appeal to Lemma 2.2 ensures that $G$ is Pettis integrable too, with $\int_\Omega G \, d\mu \subset \int_\Omega F \, d\mu$. Let $g : \Omega \to X$ be any selector of $G$. Clearly, $g$ is also a selector of $F$. We will prove that $g$ is scalarly measurable. Observe that
\[ \delta^*(x_0^*, \int_\Omega G \, d\mu) = \int_\Omega \delta^*(x_0^*, G) \, d\mu = \int_\Omega \delta^*(x_0^*, F) \, d\mu = \delta^*(x_0^*, \int_\Omega F \, d\mu) = x_0^*(x_0) = \int_\Omega (\delta^*(-x_0^*, G)) \, d\mu = -\delta^*(-x_0^*, \int_\Omega G \, d\mu). \]

It follows that $\int_\Omega G \, d\mu = \{x_0\}$. Given $x^* \in X^*$, we have $-\delta^*(-x_0^*, G) \leq \delta^*(x^*, G)$ and
\[ \int_\Omega (\delta^*(-x_0^*, G)) \, d\mu = x^*(x_0) = \int_\Omega \delta^*(x^*, G) \, d\mu, \]

hence $-\delta^*(-x_0^*, G) = \delta^*(x^*, G) \mu$-a.e. Therefore, $x^* \circ g = \delta^*(x^*, G) \mu$-a.e. and, in particular, $x^* \circ g$ is measurable. Since $x_0^* \in X^*$ is arbitrary, $g$ is scalarly measurable. Finally, an appeal to Corollary 2.3 allows us to conclude that $g$ is Pettis integrable.

In our next result we establish that in fact any Pettis integrable multi-function admits a collection of Pettis integrable selectors which are dense in it (a kind of “generalized” Castaing representation).

**Theorem 2.6.** Let $F : \Omega \to cwk(X)$ be a Pettis integrable multi-function. Then $F$ admits a collection \( \{ F_\alpha \} \subset \text{dens}(X^*, w^*) \) of Pettis integrable selectors such that
\[ F(\omega) = \{ f(\omega) : \alpha < \text{dens}(X^*, w^*) \} \quad \text{for every } \omega \in \Omega. \]

Moreover, $\int_A F \, d\mu = IS_F(A)$ for every $A \in \Sigma$. 

Corollary 2.7. □

X is convergent if and only if the series
\[
\sum_{\omega \in \Omega} x_{\alpha}^*(x) = \delta^*(x_{\alpha}^*, F(\omega)),
\]
is scalarly measurable by Lemma 2.4 and so Pettis integrable by Lemma 2.2. Then Theorem 2.5 applied to \(L_\alpha\) ensures that there is a Pettis integrable selector \(s_\alpha : \Omega \to X\) of \(L_\alpha\).

Clearly, each \(s_\alpha\) is also a selector of \(F\). We claim that
\[
F(\omega) = \text{co}\{\{s_\alpha(\omega) : \alpha < \kappa)\} \quad \text{for every } \omega \in \Omega.
\]

Indeed, fix \(\omega \in \Omega\) and set \(C := \text{co}\{\{s_\alpha(\omega) : \alpha < \kappa\} \subset F(\omega)\). Then \(C \in cwk(X)\) and
\[
\delta^*(x_{\alpha}^*, F(\omega)) = \delta^*(x_{\alpha}^*, C) \geq x_{\alpha}^*(s_\alpha(\omega)) = \delta^*(x_{\alpha}^*, F(\omega))
\]
for every \(\alpha < \kappa\). Since the set \(\{x_{\alpha}^* : \alpha < \kappa\}\) is \(\tau(X^*, X)\)-dense in \(X^*\) and the maps \(x^* \mapsto \delta^*(x^*, C)\) and \(x^* \mapsto \delta^*(x^*, F(\omega))\) are \(\tau(X^*, X)\)-continuous we obtain the equality \(\delta^*(x^*, F(\omega)) = \delta^*(x^*, C)\) for every \(x^* \in X^*\) and, therefore, \(F(\omega) = C\) as asserted.

Observe that the collection \(\{f_\alpha\}_{\alpha < \kappa}\) made up of all convex combinations of the \(s_\alpha\)'s with rational coefficients fulfills the required properties.

In order to prove the last assertion, fix \(A \in \Sigma\). Using Corollary 2.3, we obtain that
\[
\text{IS}_F(A) \subset \int_A F d\mu.
\]
On the other hand, for each \(\alpha < \kappa\), the following holds:
\[
x_{\alpha}^*(\int_A s_\alpha d\mu) = \int_A x_{\alpha}^* s_\alpha d\mu = \int_A \delta^*(x_{\alpha}^*, F) d\mu = \delta^*(x_{\alpha}^*, \int_A F d\mu),
\]
and so \(\delta^*(x_{\alpha}^*, \text{IS}_F(A)) \geq \delta^*(x_{\alpha}^*, \int_A F d\mu)\). Since \(\{x_{\alpha}^* : \alpha < \kappa\}\) is \(\tau(X^*, X)\)-dense in \(X^*\), the inequality \(\delta^*(x^*, \text{IS}_F(A)) \geq \delta^*(x^*, \int_A F d\mu)\) holds true for every \(x^* \in X^*\) and we infer that \(\int_A F d\mu \subset \text{IS}_F(A)\). Therefore \(\text{IS}_F(A) = \int_A F d\mu\) and the proof is finished.

It turns out that, when \(X^*\) is \(w^*-\)separable, the sets \(\text{IS}_F(A)\) are closed for any Pettis integrable multi-function \(F : \Omega \to cwk(X)\). The proof imitates that given in [15, Proposition 5.2] for a separable \(X\) and so we omit the details. Combining this fact with Theorem 2.6 we get the following result.

Corollary 2.7. Suppose \(X^*\) is \(w^*-\)separable. Let \(F : \Omega \to cwk(X)\) be a Pettis integrable multi-function. Then \(\int_A F d\mu = \text{IS}_F(A)\) for every \(A \in \Sigma\).

### 3. Multi-measures and Countably Additive Selectors

Given a sequence \((C_n)\) in \(cwk(X)\), the series \(\sum_n C_n\) is said to be unconditionally convergent provided that for every choice \(x_n \in C_n, n \in \mathbb{N}\), the series \(\sum_n x_n\) is unconditionally convergent in \(X\). In this case, the set
\[
\sum_n C_n := \left\{ \sum_n x_n : x_n \in C_n \text{ for all } n \in \mathbb{N} \right\}
\]
also belongs to \(cwk(X)\), see [6, Lemma 2.2]. Recall that the family \(cwk(X)\), equipped with the Hausdorff metric \(h\), is a complete metric space that can be isometrically embedded into the Banach space \(\ell_\infty(B_{X^*})\) by means of the mapping
\[
j : cwk(X) \to \ell_\infty(B_{X^*}), \quad j(C)(x^*) := \delta^*(x^*, C),
\]
see e.g. [7, Chapter II]. It is known that a series \(\sum_n C_n\) as above is unconditionally convergent if and only if the series \(\sum_n j(C_n)\) is unconditionally convergent in \(\ell_\infty(B_{X^*})\) (in this case, we have \(j(\sum_n C_n) = \sum_n j(C_n)\), cf. [6, Lemma 2.3].

**Definition 3.1.** A mapping \(M : \Sigma \to cwk(X)\) is said to be a finitely additive (resp. countably additive) multi-measure if \(M(A \cup B) = M(A) + M(B)\) whenever \(A, B \in \Sigma\) are disjoint (resp. if for every disjoint sequence \((E_n)\) in \(\Sigma\) the series \(\sum_n M(E_n)\) is unconditionally convergent and \(M(\bigcup_n E_n) = \sum_n M(E_n)\)).
Note that \( M : \Sigma \to \text{cwk}(X) \) is a finitely (resp. countably) additive multi-measure if and only if the composition \( j \circ M : \Sigma \to \ell_\infty(B_X) \) is a finitely (resp. countably) additive measure. Therefore, if for \( x^* \in X^* \) we define \( \delta^*(x^*, M) : \Sigma \to \mathbb{R} \) by \( A \mapsto \delta^*(x^*, M(A)) \), then \( M \) is a finitely additive multi-measure if and only if \( \delta^*(x^*, M) \) is finitely additive for every \( x^* \in X^* \). For countably additive multi-measures the analogue characterization is also true, see Theorem 3.4, but requires some work that we present in this section: this result, due to Costé [8] and Pallu de la Barrière [33], can be seen as the set-valued version of the well-known fact that weakly countably additive vector measures are norm countably additive (Orlicz-Pettis theorem, cf. [13, Corollary 4, p. 22]).

From a technical point of view, the novelty of our approach to Theorem 3.4 relies mostly in the way of finding “finitely additive selectors” for finitely additive multi-measures, see Theorem 3.3, via a method of “linearization” of Lipschitz functions on Banach spaces that goes back to Pelczynski [34, p. 61].

Let \( \text{Lip}_0(X^*) \) be the Banach space of all Lipschitz functions \( h : X^* \to \mathbb{R} \) satisfying \( h(0) = 0 \), equipped with the norm
\[
\|h\|_{\text{Lip}_0(X^*)} := \sup \left\{ \left| h(x_1^*) - h(x_2^*) \right| / \|x_1^* - x_2^*\| : x_1^*, x_2^* \in X^*, x_1^* \neq x_2^* \right\}.
\]

Fix an invariant mean on \( X^* \) (considered as additive abelian group), that is, a linear mapping \( \mathcal{I} : \ell_\infty(X^*) \to \mathbb{R} \) such that \( \mathcal{I}(g) \geq 0 \) whenever \( g \geq 0 \), \( \mathcal{I}(1) = 1 \) and \( \mathcal{I}(g) = \mathcal{I}(g(\cdot + x^*)) \) for every \( g \in \ell_\infty(X^*) \) and every \( x^* \in X^* \); cf. [25, Theorem 17.5]. It is known that we can define an operator \( P : \text{Lip}_0(X^*) \to X^{**} \) by the formula
\[
\langle P(h), x^* \rangle := \mathcal{I}(h(\cdot + x^*) - h(\cdot)), \quad h \in \text{Lip}_0(X^*), \quad x^* \in X^*,
\]
cf. [3, Proposition 7.5].

**Lemma 3.2.** Let \( C \in \text{cwk}(X) \). Then \( \delta^*(\cdot, C) \in \text{Lip}_0(X^*) \) and \( P(\delta^*(\cdot, C)) \in C \).

**Proof.** The first assertion is clear, since
\[
|\delta^*(x_1^*, C) - \delta^*(x_2^*, C)| \leq \|x_1^* - x_2^*\| \cdot \sup \{ \|x\| : x \in C \} \quad \text{for every } x_1^*, x_2^* \in X^*.
\]
The proof of the second assertion is by contradiction. Suppose that \( P(\delta^*(\cdot, C)) \notin C \). Since \( C \) is a convex \( w^* \)-closed subset of \( X^{**} \), the Hahn-Banach separation theorem guarantees the existence of some \( x^* \in X^* \) such that
\[
(1) \quad \langle P(\delta^*(\cdot, C)), x^* \rangle > \sup \{ x^*(x) : x \in C \} = \delta^*(x^*, C).
\]
On the other hand, we have \( \delta^*(y^* + x^*, C) - \delta^*(y^*, C) \leq \delta^*(x^*, C) \) for every \( y^* \in X^* \), and the properties of \( \mathcal{I} \) yield
\[
P(\delta^*(\cdot, C)) = \mathcal{I}(\delta^*(\cdot + x^*, C) - \delta^*(\cdot, C)) \leq \mathcal{I}(\delta^*(x^*, C)) = \delta^*(x^*, C),
\]
which contradicts (1). The proof is over. \( \Box \)

We are now ready to deal with the aforementioned results about multi-measures.

**Theorem 3.3** ([20], [8] and [33]). Let \( M : \Sigma \to \text{cwk}(X) \) be a finitely additive multi-measure. Then there is a finitely additive measure \( m : \Sigma \to X \) such that \( m(A) \in M(A) \) for every \( A \in \Sigma \).

**Proof.** Lemma 3.2 ensures that \( \delta^*(\cdot, M(A)) \in \text{Lip}_0(X^*) \) and
\[
m(A) := P(\delta^*(\cdot, M(A))) \in M(A) \quad \text{for every } A \in \Sigma.
\]
Since \( M \) is a finitely additive multi-measure and \( P \) is linear, \( m \) is finitely additive. \( \Box \)

For a given \( x^* \in B_{X^{**}} \), let \( e_{x^*} \) denote the element of \( B_{cwk(B_X^{**})} \) defined by the formula
\[
e_{x^*}(\varphi) := \varphi(x^*).
\]

**Theorem 3.4** (Costé-Pallu de la Barrière). Let \( M : \Sigma \to \text{cwk}(X) \) be a mapping. The following statements are equivalent:
(i) $M$ is a countably additive multi-measure.
(ii) $\delta^*(x^*, M)$ is countably additive for every $x^* \in X^*$.
(iii) $\delta^*(x^*, M)$ is countably additive for every $x^* \in X^*$ and there is a countably additive measure $m : \Sigma \to X$ such that $m(A) \in M(A)$ for every $A \in \Sigma$.

Proof. The implication (i)$\Rightarrow$(ii) follows from the fact that $\delta^*(x^*, M) = \langle e_{x^*}, j \circ M \rangle$ for every $x^* \in B_{X^*}$.

Let us prove (ii)$\Rightarrow$(iii). By Theorem 3.3 there is a finitely additive measure $m : \Sigma \to X$ such that $m(A) \in M(A)$ for every $A \in \Sigma$. We claim that $m$ is countably additive. To prove that it suffices to show that the composition $x^* \circ m$ is countably additive for every $x^* \in X^*$ and then appeal to the Orlicz-Pettis theorem, see [13, Corollary 4, p. 22]. Given $x^* \in X^*$, we have $-\delta^*(-x^*, M(A)) \leq (x^* \circ m)(A) \leq \delta^*(x^*, M(A))$ for every $A \in \Sigma$. Since both $-\delta^*(-x^*, M)$ and $\delta^*(x^*, M)$ are countably additive and $x^* \circ m$ is finitely additive, it follows that $x^* \circ m$ is countably additive, as claimed.

To finish we prove (iii)$\Rightarrow$(i). We will prove that the finitely additive measure $\nu := j \circ M : \Sigma \to \ell_\infty(B_{X^*})$ is countably additive. The proof is divided into two cases.

Particular case. Suppose $m(A) = 0$ for every $A \in \Sigma$. Take a disjoint sequence $(A_n)$ in $\Sigma$. We will show first that the series $\sum_n \nu(A_n)$ is unconditionally convergent. This is equivalent to saying that the series of sets $\sum_n M(A_n)$ is unconditionally convergent. Fix $x_n \in M(A_n)$ for every $n \in \mathbb{N}$, and take a sequence $n_1 < n_2 < \ldots$ in $\mathbb{N}$. Define $s_k = \sum_{i=1}^k x_{n_i}$ for every $k \in \mathbb{N}$. Note that

$$s_k = s_k + 0 \in \sum_{i=1}^k M(A_{n_i}) + M\left(\Omega \setminus \bigcup_{i=1}^k A_{n_i}\right) = M(\Omega) \quad \text{for every } k \in \mathbb{N}.$$ 

On the other hand, for each $x^* \in X^*$ the series $\sum_{i=1}^\infty x^*(x_{n_i})$ is convergent. Indeed, it suffices to bear in mind that

$$\sum_{i=1}^\infty |x^*(x_{n_i})| \leq \sum_{i=1}^\infty |\delta^*(x^*, M(A_{n_i}))| + \sum_{i=1}^\infty |\delta^*(-x^*, M(A_{n_i}))| < +\infty.$$ 

This ensures that the sequence $(s_k)$ has at most one weak cluster point in $X$. Since $(s_k)$ is contained in the weakly compact set $M(\Omega)$, it follows that the series $\sum_{i=1}^\infty x_{n_i}$ is weakly convergent. As the sequence $n_1 < n_2 < \ldots$ is arbitrary, the Orlicz-Pettis theorem (cf. [13, Corollary 4, p. 22]) ensures that the series $\sum_{n=1}^\infty x_n$ is unconditionally convergent. This proves that the series $\sum_n \nu(A_n)$ converges unconditionally in $\ell_\infty(B_{X^*})$.

We claim now that $\sum_{n=1}^\infty \nu(A_n) = \nu(\bigcup_{n=1}^\infty A_n)$. Indeed, for each $x^* \in B_{X^*}$ we have

$$\left(\sum_{n=1}^\infty \nu(A_n)(x^*)\right) = \lim_{N \to \infty} \sum_{n=1}^N \nu(A_n)(x^*) = \lim_{N \to \infty} \sum_{n=1}^N \delta^*(x^*, M(A_n)) = \delta^*(x^*, M(\bigcup_{n=1}^\infty A_n)) = \nu\left(\bigcup_{n=1}^\infty A_n\right)(x^*).$$ 

The proof of the Particular case is finished.

General case. Define the mapping

$$M' : \Sigma \to cwk(X), \quad M'(A) = -m(A) + M(A).$$ 

It is clear that $\delta^*(x^*, M') = -(x^* \circ m + \delta^*(x^*, M))$ for every $x^* \in X^*$. Note also that $0 \in M'(A)$ for every $A \in \Sigma$. The Particular case already proved ensures that the mapping $\nu' := j \circ M' : \Sigma \to \ell_\infty(B_{X^*})$ is a countably additive measure. On the other hand, the mapping $\nu'' : \Sigma \to \ell_\infty(B_{X^*})$ given by $\nu''(A)(x^*) := x^*(m(A))$ is obviously a countably additive measure. It follows that $\nu = \nu' + \nu''$ is countably additive, as required.

For further information on the theory of multi-measures, we refer the reader to [23, Section 7], [27, Chapter 19] and the references therein.
4. Characterization of Pettis Integrability for Multi-Functions

The aim of this section is to discuss the validity of Theorem A in the introduction within the setting of non-separable Banach spaces. Note that Corollary 2.3 gives us the extension to the non-separable case of (i) \(\Rightarrow\) (iii) in Theorem A.

With the help of the results about multi-measures isolated in Section 3 we start by proving Theorem 4.1 below that extends to the non-separable case the implication (i) \(\Rightarrow\) (iii) in Theorem A, see (d) \(\Rightarrow\) (e) in [15, Theorem 5.4]. Given \(F : \Omega \to cwk(X)\) we write

\[
W_F := \{\delta^*(x^*, F) : x^* \in B_{X^*}\} \subset \mathbb{R}^\Omega.
\]

Recall that a family \(\mathcal{H}\) of real-valued integrable functions defined on \(\Omega\) is said to be uniformly integrable if it is bounded for \(\|\cdot\|_1\) and for each \(\varepsilon > 0\) there is \(\delta > 0\) such that \(\sup_{h \in \mathcal{H}} \int_{|h| < \delta} |d\mu| \leq \varepsilon\) whenever \(\mu(E) \leq \delta\).

**Theorem 4.1.** Let \(F : \Omega \to cwk(X)\) be a Pettis integrable multi-function. Define the indefinite Pettis integral of \(F\) by

\[
I_F : \Sigma \to cwk(X), \quad I_F(A) := \int_A F d\mu.
\]

Then:

(i) \(I_F\) is a countably additive multi-measure.
(ii) \(W_F\) is uniformly integrable.

**Proof.** Clearly, \(\delta^*(x^*, I_F)\) is countably additive for every \(x^* \in X^*\) and we can apply Theorem 3.4 to conclude that \(I_F\) is a countably additive multi-measure. This proves (i).

We prove now statement (ii). The composition \(\nu := j \circ I_F : \Sigma \to \ell_{\infty}(B_{X^*})\) is a countably additive vector measure that vanishes on all \(\mu\)-null sets. Hence \(\nu\) is \(\mu\)-continuous, that is, \(\lim_{\mu(A) \to 0} \|\nu\|(A) = 0\) (cf. [13, Theorem 1, p. 10]). On the other hand, observe that \([e_{x^*}, \nu] (A) = \int_A \delta^*(x^*, F) d\mu\) for every \(x^* \in B_{X^*}\) and every \(A \in \Sigma\). In view of the above, the uniform integrability of \(W_F\) now follows from the fact that

\[
\|\nu\|(A) \geq \sup_{x^* \in B_{X^*}} |[e_{x^*}, \nu]|(A) = \sup_{x^* \in B_{X^*}} \int_A |\delta^*(x^*, F)| d\mu
\]

for every \(A \in \Sigma\).

We turn our attention now to the implication (iii) \(\Rightarrow\) (i) in Theorem A for the non separable case: the proof below is inspired by some of the ideas in [15, Theorems 3.9 and 5.4]. We say that a Banach space \(X\) has the Scalarly Measurable Selector Property with respect to \(\mu\), shortly \(\mu\)-SMSP, if every scalarly measurable multi-function \(F : \Omega \to cwk(X)\) has a scalarly measurable selector.

**Theorem 4.2.** Suppose \(X\) has the \(\mu\)-SMSP. Let \(F : \Omega \to cwk(X)\) be a scalarly measurable multi-function such that every scalarly measurable selector of \(F\) is Pettis integrable. Then \(F\) is Pettis integrable.

**Proof.** For any fixed \(A \in \Sigma\) the set \(\overline{TS_F(A)}\) is closed and convex. We prove now that \(\overline{TS_F(A)} \subset cwk(X)\). By James’ theorem (cf. [17, §5]) we only have to prove that every \(x^* \in X^*\) attains its supremum on \(\overline{TS_F(A)}\). Fix \(x^* \in X^*\) and consider the multi-function

\[
G_{x^*} : \Omega \to cwk(X), \quad G_{x^*}(\omega) := \{x \in F(\omega) : x^*(x) = \delta^*(x^*, F(\omega))\}.
\]

Since \(G_{x^*}\) is scalarly measurable (by Lemma 2.4) and \(X\) has the \(\mu\)-SMSP, there is a scalarly measurable selector \(g_{x^*}\) of \(G_{x^*}\). In particular, \(g_{x^*}\) is a selector of \(F\) and so it is Pettis integrable. Hence \(\delta^*(x^*, F) = x^* \circ g_{x^*}\) is integrable. By the very definition, we have \(\int_A g_{x^*} d\mu \in IS_F(A)\). We claim that

\[
\sup\{x^*(x) : x \in IS_F(A)\} = x^*\left(\int_A g_{x^*} d\mu\right).
\]
Indeed, notice that for each Pettis integrable selector $f$ of $F$ we have
\[
x^* \left( \int_A g_{x^*} \, d\mu \right) = \int_A x^* \circ g_{x^*} \, d\mu = \int_A \delta^*(x^*, F) \, d\mu \geq \int_A x^* \circ f \, d\mu = x^* \left( \int_A f \, d\mu \right),
\]

hence
\[
\sup \{ x^*(x) : x \in \overline{IS}_F(A) \} = \sup \{ x^*(x) : x \in IS_F(A) \} = x^* \left( \int_A g_{x^*} \, d\mu \right).
\]

This proves that $\overline{IS}_F(A)$ is weakly compact. Moreover, the previous equality can be read as $\delta^*(x^*, \overline{IS}_F(A)) = \int_A \delta^*(x^*, F) \, d\mu$. It follows that $F$ is Pettis integrable. \hfill \Box

Recall that the Banach space $X$ is said to have the $\mu$-Pettis Integral Property (shortly $\mu$-PIP) if every scalarly measurable and scalarly bounded function $f : \Omega \to X$ is Pettis integrable. Here $f : \Omega \to X$ is said to be scalarly bounded if there is $M > 0$ such that for each $x^* \in B_{X^*}$, we have $|x^* \circ f| \leq M$ $\mu$-a.e. (the exceptional set depending on $x^*$). Equivalently, $X$ has the $\mu$-PIP if and only if the Pettis integrability of any function $f : \Omega \to X$ is equivalent to the fact that the family
\[
\mathcal{Z}_f = \{ x^* \circ f : x^* \in B_{X^*} \} \subset \mathbb{R}^\Omega
\]
is uniformly integrable.

**Corollary 4.3.** Suppose $X$ has the $\mu$-SMSP and the $\mu$-PIP. Let $F : \Omega \to \text{cwk}(X)$ be a multi-function. Then $F$ is Pettis integrable if and only if $W^c_F$ is uniformly integrable.

**Proof.** It only remains to prove the “if” part. Observe that $F$ is scalarly measurable. Each scalarly measurable selector $f$ of $F$ satisfies $-\delta^*(-x^*, F) \leq x^* \circ f \leq \delta^*(x^*, F)$ for all $x^* \in B_{X^*}$. Since $W^c_F$ is uniformly integrable, the same holds for $\mathcal{Z}_f$ and thus $f$ is Pettis integrable (because $X$ has the $\mu$-PIP). The result now follows from Theorem 4.2. \hfill \Box

The Banach space $X$ has the PIP if it has the $\mu$-PIP for any complete probability measure $\mu$. The class of Banach spaces with the PIP is very large and contains, for instance, all spaces having Corson’s property (C), see [42, Theorem 5.2-4], hence all weakly Lindelöf Banach spaces and all Banach spaces with $w^*$-angelic dual [35]. Recall that a topological space $T$ is said to be angelic if each relatively countably compact set $C \subset T$ is relatively compact and, moreover, each point in the closure of $C$ is the limit of a sequence in $C$.

The following cardinal number will be used in several examples that follow:
\[
\kappa(\mu) = \min \{ \text{card}(E) : E \subset \Sigma, \mu(E) = 0 \text{ for every } E \in \mathcal{E}, \mu^*(\cup \mathcal{E}) > 0 \},
\]
defined if there exist such infinite families $\mathcal{E}$ (this happens, for instance, if $\mu$ is not purely atomic). Here $\mu^*$ denotes the outer measure induced by $\mu$. Notice that $\kappa(\mu) \geq \omega_1$. We point out that the intersection of less than $\kappa(\mu)$ elements of $\Sigma$ also belongs to $\Sigma$, cf. [38, Lemma 4.4]. When $\kappa(\mu)$ cannot be defined, the intersection of any family of measurable sets is measurable and all our results involving $\kappa(\mu)$ are true without the restrictions on the cardinalities or density characters appearing in their statement. It is well known (cf. [40]) that Martin’s Axiom implies the statement
\[
\kappa(\text{Lebesgue measure on } [0, 1]) = \mathcal{C} \text{ (Axiom } M)\]

The Banach space $X$ has both the $\mu$-SMSP and the PIP in each of the following cases:
\begin{itemize}
  \item $X$ is separable.
  \item $X$ is reflexive, Theorem 5.1.
  \item $(X^*, w^*)$ is angelic and $\text{dens}(X^*, w^*) \leq \kappa(\mu)$, Example 5.5.
  \item $X = Y^*$ has property (C) and $\text{dens}(Y) \leq \kappa(\mu)$, Example 5.6.
\end{itemize}
On the other hand, we will also see that $X$ has the $\mu$-SMSP whenever $X^*$ is $w^*$-separable, Theorem 5.15. However, such an $X$ does not have the $\mu$-PIP in general. Indeed, Fremlin and Talagrand [18] showed that $\ell_\infty(\mathbb{N})$ fails the $\mu$-PIP for certain pathological measure $\mu$. They also proved that, at least under Axiom M, if $B_{X^*}$ is $w^*$-separable for some equivalent norm on $X$ (equivalently, $X$ is isomorphic to a subspace of $\ell_\infty(\mathbb{N})$), then $X$ has the PIP with respect to any perfect measure (for instance, a Radon finite measure on a topological space), cf. [42, Theorems 6-1-2 and 6-1-3].

We end up this section turning our attention to the following question, thoroughly studied in [5] and [6] within the setting of separable Banach spaces:

What is the relationship between the Pettis integrability of the multi-function $F : \Omega \to cwk(X)$ and that of the single-valued composition $j \circ F : \Omega \to \ell_\infty(B_{X^*})$?

As in the separable case, see [6, Proposition 3.5], $F$ is Pettis integrable whenever $j \circ F$ is. The proof of this fact given here is more direct.

**Proposition 4.4.** Let $F : \Omega \to cwk(X)$ be a multi-function such that $j \circ F$ is Pettis integrable. Then $F$ is Pettis integrable and

$$j(I_F(A)) = \int_A j \circ F \, d\mu \quad \text{for every } A \in \Sigma.$$ 

**Proof.** Since $j \circ F$ is Pettis integrable, the composition $(e_{x^*}, j \circ F) = \delta^*(x^*, F)$ is integrable for every $x^* \in B_{X^*}$. Fix $A \in \Sigma$. The Pettis integrability of $j \circ F$ and the Hahn-Banach separation theorem ensure that

$$\int_A j \circ F \, d\mu \in \mu(A) \cdot \co((j \circ F)(A)),$$

cf. [13, proof of Corollary 8, p. 48]. Since $j(cwk(X))$ is a closed convex cone, we conclude that $\int_A j \circ F \, d\mu = j(C_A)$ for some $C_A \in cwk(X)$. Then

$$\int_A \delta^*(x^*, F) \, d\mu = \int_A \langle e_{x^*}, j \circ F \rangle \, d\mu = \langle e_{x^*}, \int_A j \circ F \, d\mu \rangle = \delta^*(x^*, C_A)$$

for every $x^* \in B_{X^*}$. This shows that $F$ is Pettis integrable, with $j(I_F(A)) = \int_A j \circ F \, d\mu$ for every $A \in \Sigma$. $\square$

It is known that the converse of Proposition 4.4 does not hold in general even for separable Banach spaces, see [5, Theorem 2.1]. However, it is valid under some additional assumptions on the given multi-function.

**Proposition 4.5.** Let $F : \Omega \to cwk(X)$ be a multi-function such that $(j \circ F)(\Omega)$ is contained in a subspace of $\ell_\infty(B_{X^*})$ having $w^*$-angelic dual (this happens, for instance, if $F(\Omega)$ is separable for the Hausdorff distance). The following statements are equivalent:

1. $F$ is Pettis integrable;
2. $W_F$ is uniformly integrable;
3. $j \circ F$ is Pettis integrable.

**Proof.** The implication (i)⇒(ii) follows from Theorem 4.1 and (iii)⇒(i) from Proposition 4.4. Let us prove (ii)⇒(iii): let $Y \subset \ell_\infty(B_{X^*})$ be a subspace containing $(j \circ F)(\Omega)$ such that $Y^*$ is $w^*$-angelic. Notice that the set $B := \{e_{x^*} \mid x^* \in B_{X^*}\} \subset B_{Y^*}$ is norming. The desired conclusion now follows by applying [6, Lemma 3.3] to the $Y$-valued function $j \circ F$, see the comments in [6, p. 552]. $\square$

Recall that a convex, closed, bounded, non-empty set $C \subset X$ is norm compact if and only if the real-valued mapping given by $x^* \mapsto \delta^*(x^*, C)$ is $w^*$-continuous on $B_{X^*}$, cf. [31, Section 7]. Thus $j(ck(X)) \subset C(B_{X^*}) = C(B_{X^*}, w^*)$.
Proposition 4.6. Suppose $X^*$ is $w^*$-angelic. Let $F : \Omega \to \text{cwk}(X)$ be a multi-function with norm compact values such that $W_F$ is uniformly integrable. Then $F$ is Pettis integrable and $I_F(A)$ is norm compact for every $A \in \Sigma$.

Proof. Fix $A \in \Sigma$. We claim that the mapping $\varphi^F_A : X^* \to \mathbb{R}$ given by $\varphi^F_A(x^*) = \int_A \delta^*(x^*, F) \, d\mu$ is $w^*$-continuous when restricted to $B_{X^*}$. Indeed, fix $B \subset B_{X^*}$ and take $x^* \in \overline{B}_{X^*}$. Since $(X^*, w^*)$ is angelic, there is a sequence $(x^*_n)$ in $B$ converging to $x^*$ in the $w^*$-topology. Given $\omega \in \Omega$, the set $F(\omega)$ is norm compact and so the mapping $\delta^*(\cdot, F(\omega))$ is $w^*$-continuous on $B_{X^*}$, hence $\delta^*(x^*_n, F(\omega)) \to \delta^*(x^*, F(\omega))$ as $n \to \infty$.

Since $W_F$ is uniformly integrable, an appeal to Vitali’s convergence theorem ensures that

$$\varphi^F_A(x^*_n) = \int_A \delta^*(x^*_n, F) \, d\mu \to \int_A \delta^*(x^*, F) \, d\mu = \varphi^F_A(x^*) \quad \text{as} \quad n \to \infty.$$  

As $x^* \in \overline{B}_{X^*}$ is arbitrary, we conclude that $\varphi^F_A(B_{X^*}) \subset \overline{\varphi^F_A(B)}$. Since this inclusion holds for any set $B \subset B_{X^*}$, the restriction $\varphi^F_A|_{B_{X^*}}$ is $w^*$-continuous, as claimed. Similarly, $\varphi^F_A|_{B_{X^*}}$ is $w^*$-continuous for every $n \in \mathbb{N}$. Bearing in mind that $\varphi^F_A$ is convex, an appeal to the Banach-Dieudonné theorem (cf. [16, Theorem 4.44]) ensures that $\varphi^F_A$ is $w^*$-lower semicontinuous. By [7, Theorem II-16], there is a convex, closed, bounded, non-empty set $C \subset X$ such that $\varphi^F_A(x^*) = \delta^*(x^*, C)$ for every $x^* \in X^*$. The $w^*$-continuity of $\varphi^F_A|_{B_{X^*}}$ guarantees that $C$ is norm compact and the proof is over. \hfill \Box

5. MEASURABLE SELECTORS

5.1. Scalarly measurable selectors. The first measurable selection results of this subsection follow from the existence of scalarly measurable selectors for Pettis integrable $cwk(X)$-valued functions, Theorem 2.5 above.

Theorem 5.1. If $X$ is reflexive, then it has the $\mu$-SMSP.

Proof. Let $F : \Omega \to \text{cwk}(X)$ be a scalarly measurable multi-function. Since

$$\{\delta^*(x^*, F) : x^* \in X^*, \|x^*\| = 1\}$$

is a pointwise bounded family of measurable functions, we can find a countable partition $E_1, E_2, \ldots$ of $\Omega$ in $\Sigma$ and a sequence $(M_n)$ of positive real numbers such that, for each $n \in \mathbb{N}$ and each $x^* \in X^*$ with $\|x^*\| = 1$, we have $|\delta^*(x^*, F)|_{E_n} \leq M_n$ $\mu$-a.e. (cf. [32, Proposition 3.1]). Fix $n \in \mathbb{N}$ and consider the (constant) Pettis integrable multi-function $H_n : E_n \to \text{cwk}(X)$ given by $H_n(\omega) := M_n B_X$. Observe that for each $x^* \in X^*$ we have $\delta^*(x^*, F|_{E_n}) \leq \delta^*(x^*, H_n)$ $\mu$-a.e. From Lemma 2.2 it follows that $F|_{E_n}$ is Pettis integrable. By Theorem 2.5, we know that $F|_{E_n}$ admits a scalarly measurable selector $f_n : E_n \to X$. Define $f : \Omega \to X$ by $f(\omega) := f_n(\omega)$ if $\omega \in E_n$, $n \in \mathbb{N}$. Clearly, $f$ is a scalarly measurable selector of $F$. \hfill \Box

Theorem 5.2. Suppose $X^*$ is $w^*$-angelic. Then every scalarly measurable multi-function $F : \Omega \to \text{cwk}(X)$ admits a scalarly measurable selector.

Proof. Again, since $W_F$ is a pointwise bounded family of measurable functions, there is a countable partition $E_1, E_2, \ldots$ of $\Omega$ in $\Sigma$ and a sequence $(M_n)$ of positive real numbers such that, for each $n \in \mathbb{N}$ and each $x^* \in B_{X^*}$, we have $|\delta^*(x^*, F)|_{E_n} \leq M_n$ $\mu$-a.e. Given $n \in \mathbb{N}$, the previous inequality ensures that the family $W_F|_{E_n}$ is uniformly integrable and Proposition 4.6 can be applied to conclude that $F|_{E_n}$ is Pettis integrable. The proof finishes as in Theorem 5.1. \hfill \Box

At this point it is convenient to introduce the following terminology. Given a topological space $T$, we denote by $k(T)$ the collection of all compact non-empty subsets of $T$. Let $\mathcal{M}$ be a non-empty family of closed subsets of $T$. We say that a multi-function $F : \Omega \to k(T)$ is $\mathcal{M}$-measurable if $\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} \in \Sigma$ for every $M \in \mathcal{M}$. Clearly, with this
Lemma 5.3. Let $T$ be a topological space and $M$ a non-empty family of closed subsets of $T$. Let $\gamma < \kappa(\mu)$ and, for each $\alpha < \gamma$, let $F_\alpha : \Omega \to k(T)$ be a $M$-measurable multi-function. Suppose $F_\beta(\omega) \supset F_\alpha(\omega)$ for every $\beta < \alpha < \gamma$ and every $\omega \in \Omega$. Then:

(i) For each $\omega \in \Omega$, the set $F(\omega) := \bigcap_{\alpha < \gamma} F_\alpha(\omega)$ is compact and non-empty.

(ii) The multi-function $F : \Omega \to k(T)$ is $M$-measurable.

Proof. Given $\omega \in \Omega$, the net of compact non-empty sets $(F_\alpha(\omega))_{\alpha < \gamma}$ is decreasing and so it has compact non-empty intersection. In order to prove the second assertion, take $M \in M$ and observe that, since $(F_\alpha(\omega) \cap M)_{\alpha < \gamma}$ is a decreasing net of compact sets, we have

$$\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} = \bigcap_{\alpha < \gamma} \{\omega \in \Omega : F_\alpha(\omega) \cap M \neq \emptyset\}.$$  

The $M$-measurability of each $F_\alpha$ ensures that $\{\omega \in \Omega : F_\alpha(\omega) \cap M \neq \emptyset\} \in \Sigma$. Since $\text{card}(\gamma) < \kappa(\mu)$, it follows that $\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} \in \Sigma$. \hfill \Box

Our approach to the next theorem is inspired somehow by some of the ideas in the original proof of Valadier’s result [43] saying that Banach spaces with $w^*$-separable dual always have the $\mu$-SMSP (Theorem 5.15 below).

Theorem 5.4. Suppose there is a set $\Gamma \subset X^*$ satisfying the following properties:

(i) $\text{card}(\Gamma) \leq \kappa(\mu)$.

(ii) $\Gamma$ separates the points of $X$.

(iii) A function $f : \Omega \to X$ is scalarly measurable if and only if $x^* \circ f$ is measurable for every $x^* \in \Gamma$.

Then $X$ has the $\mu$-SMSP.

Proof. Enumerate $\Gamma = \{x^*_\alpha : \alpha < \text{card}(\Gamma)\}$. Fix a scalarly measurable multi-function $F : \Omega \to \text{cwk}(X)$. We divide the proof of the existence of a scalarly measurable selector of $F$ into several steps.

Step 1. Define $F_0 := F$. We will construct by transfinite induction a family of scalarly measurable multi-functions $F_\alpha : \Omega \to \text{cwk}(X)$, with $\alpha < \text{card}(\Gamma)$, such that

$$F_\alpha(\omega) = \bigcap_{\beta < \alpha} \{x \in F_\beta(\omega) : x^*_\beta(x) = \delta^*(x^*_\beta, F_\beta(\omega))\} \quad \text{for all} \ \omega \in \Omega$$

for every $0 < \alpha < \text{card}(\Gamma)$. To this end, assume that $0 < \gamma < \text{card}(\Gamma)$ and that we have already constructed a family $(F_\alpha)_{\alpha < \gamma}$ of scalarly measurable multi-functions satisfying (2) for every $0 < \alpha < \gamma$. Given $\alpha < \gamma$, Lemma 2.4 applies to conclude that the multi-function $G_\alpha : \Omega \to \text{cwk}(X)$ given by

$$G_\alpha(\omega) := \{x \in F_\alpha(\omega) : x^*_\alpha(x) = \delta^*(x^*_\alpha, F_\alpha(\omega))\}$$

is scalarly measurable. Observe that $G_\beta(\omega) \supset G_\alpha(\omega)$ for every $\beta < \alpha < \gamma$ and every $\omega \in \Omega$. Since $\gamma < \text{card}(\Gamma) \leq \kappa(\mu)$, Lemma 5.3 allows us to define a scalarly measurable multi-function $F_\gamma : \Omega \to \text{cwk}(X)$ by the formula $F_\gamma(\omega) := \bigcap_{\alpha < \gamma} G_\alpha(\omega)$. Obviously, $F_\gamma$ satisfies (2) by construction.

Step 2. Given $\omega \in \Omega$, the net of weakly compact non-empty sets $(F_\alpha(\omega))_{\alpha < \text{card}(\Gamma)}$ is decreasing and so $\bigcap_{\alpha < \text{card}(\Gamma)} F_\alpha(\omega) \neq \emptyset$. In fact, this intersection contains only one point. Indeed, if $x_1, x_2 \in \bigcap_{\alpha < \text{card}(\Gamma)} F_\alpha(\omega)$, then

$$x^*_\beta(x_1) = \delta^*(x^*_\beta, F_\beta(\omega)) = x^*_\beta(x_2)$$
for every $\beta < \text{card}(\Gamma)$, and the fact that $\Gamma$ separates the points of $X$ implies $x_1 = x_2$. Therefore, there is a function $f : \Omega \to X$ such that

$$\bigcap_{\alpha < \text{card}(\Gamma)} F_\alpha(\omega) = \{f(\omega)\} \quad \text{for every } \omega \in \Omega.$$

**Step 3.** Clearly, $f$ is a selector of $F$. By assumption, in order to prove that $f$ is scalarly measurable we only have to check that $x_0^* \circ f$ is measurable for every $\beta < \text{card}(\Gamma)$. Indeed, take $\beta < \alpha < \text{card}(\Gamma)$. Then $f(\omega) \in F_\alpha(\omega)$ and therefore $x_0^*(f(\omega)) = \delta^\ast(x_0^*, F_\beta(\omega))$ for every $\omega \in \Omega$. Since $F_\beta$ is scalarly measurable, we conclude that $x_0^* \circ f$ is measurable. The proof is over. 

A well known result of Edgar, see [14, Theorem 2.3], states that the Baire $\sigma$-algebra of a locally convex space endowed with its weak topology is exactly the $\sigma$-algebra generated by all the elements of the topological dual. In particular, if $\Gamma \subset X^*$ is a set separating the points of $X$ and $\text{Baire}(X, \sigma(X, \Gamma))$ denotes the topology on $X$ of pointwise convergence on $\Gamma$, then $\text{Baire}(X, \sigma(X, \Gamma))$ is just the $\sigma$-algebra on $X$ generated by $\Gamma$. Thus, condition (iii) in Theorem 5.4 is equivalent to “$f$ is Baire($X, \sigma(X, \Gamma)$)-measurable”. Bearing this in mind, observe that Theorem 5.4 ensures that $X$ has the $\mu$-$\text{SMSP}$ in the following two cases:

**Example 5.5.** ($X^*, w^*$) is angelic and $\text{dens}(X^*, w^*) \leq \kappa(\mu)$. By a result of Gulisashvili [21], when ($X^*, w^*$) is angelic, the equality $\text{Baire}(X, \sigma(X, \Gamma)) = \text{Baire}(X, w)$ holds for any set $\Gamma \subset X^*$ separating the points of $X$. A wide class of spaces having $w^*$-angelic dual is that of weakly Lindelöf determined (WLD) Banach spaces. This class contains all weakly compactly generated spaces (cf. [16, Chapters 11 and 12]) and for every WLD space $X$ the equality $\text{dens}(X^*, w^*) = \text{dens}(X)$ holds. In particular, any weakly compactly generated Banach space with density character less than or equal to $\omega_1$ has the $\mu$-$\text{SMSP}$. For instance, this applies to $c_0(\omega_1)$, separable Banach spaces, etc.

**Example 5.6.** $X = Y^*$ has property (C) and $\text{dens}(Y) \leq \kappa(\mu)$. Indeed, any norm dense set $\Gamma \subset Y$ separates the points of $X$ and satisfies $\text{Baire}(X, \sigma(X, \Gamma)) = \text{Baire}(X, w)$. On the other hand, since $X$ is a dual space having property (C), the equality $\text{Baire}(X, w) = \text{Baire}(X, w)$ holds, see [39, Corollary 3.10].

Next three lemmas are needed to prove Theorem 5.10.

**Lemma 5.7.** Let $A \subset \text{cwk}(X)$ and $x_0^* \in X^*$ satisfying $\inf x_0^*(A) < b < \sup x_0^*(A)$ for some $b \in \mathbb{R}$. Let $x \in A$ such that $x_0^*(x) \geq b$. Then for every $\varepsilon > 0$ there is $y \in A$ such that $\|x - y\| \leq \varepsilon$ and $x_0^*(y) \in [b, \sup x_0^*(A)] \cap \mathbb{Q}$.

**Proof.** Since $A \subset \text{cwk}(X)$, we have $x_0^*(A) = [\inf x_0^*(A), \sup x_0^*(A)]$. There are two possibilities:

**Case 1.** Suppose $x_0^*(x) < \inf x_0^*(A)$. Fix $z \in A$ such that $x_0^*(z) = \sup x_0^*(A)$ and consider the mapping $\varphi : [0, 1] \to [x_0^*(x), \sup x_0^*(A)]$ given by $\varphi(\theta) := x_0^*(\theta z + (1 - \theta)x)$. We can choose $0 < \theta < \min\{\varepsilon/\|x - z\|, 1\}$ such that $\varphi(\theta) \in \mathbb{Q}$. Then the vector $y := \theta z + (1 - \theta)x$ satisfies the required properties.

**Case 2.** Suppose $x_0^*(x) = \sup x_0^*(A)$. Take $z \in A$ such that $x_0^*(z) = b$ and consider now the mapping $\varphi : [0, 1] \to [b, \sup x_0^*(A)]$ given by $\varphi(\theta) := x_0^*(\theta z + (1 - \theta)x)$. Choose $0 < \theta < \min\{\varepsilon/\|x - z\|, 1\}$ such that $\varphi(\theta) \in \mathbb{Q}$. Then $y := \theta z + (1 - \theta)x$ works. 

**Lemma 5.8** ([43, Lemma 3] or [7, Proposition 1-24]). Let $C \subset \text{cwk}(X)$, $x_0^* \in X^*$ and $\alpha \in \mathbb{R}$. Suppose $H := \{x \in X : x_0^*(x) = \alpha\}$ intersects $C$. Then $C \cap H \subset \text{cwk}(X)$ and

$$\delta^\ast(x^*, C \cap H) = \inf\{\delta^\ast(x^* - \lambda x_0^*, C) + \lambda \alpha : \lambda \in \mathbb{Q}\} \quad \text{for every } x^* \in X^*.$$  

**Lemma 5.9.** Let $F : \Omega \to \text{cwk}(X)$ be a scalarly measurable multi-function and consider a measurable function $h : \Omega \to \mathbb{R}$. Fix $x_0^* \in X^*$ and write

$$L(\omega) := \{x \in X : x_0^*(x) \geq h(\omega)\} \quad \text{for every } \omega \in \Omega.$$
Then \( E := \{ \omega \in \Omega : F(\omega) \cap L(\omega) \neq \emptyset \} \in \Sigma \) and the multi-function \( G : E \to cwk(X) \), \( G(\omega) := F(\omega) \cap L(\omega) \), is scalarly measurable.

**Proof.** Clearly, the set \( E = \{ \omega \in \Omega : \delta^*(x^*_0, F(\omega)) \geq h(\omega) \} \) belongs to \( \Sigma \). Note that 
\[-\delta^*(-x^*_0, F(\omega)) = \inf x^*_0(F(\omega)) \text{ for every } \omega \in \Omega.\]
The sets
\[E_1 := \{ \omega \in E : \inf x^*_0(F(\omega)) \geq h(\omega) \}\]
\[E_2 := \{ \omega \in E : \sup x^*_0(F(\omega)) = h(\omega) \}\]
\[E_3 := \{ \omega \in E : \inf x^*_0(F(\omega)) < h(\omega) < \sup x^*_0(F(\omega)) \}\]
belong to \( \Sigma \) and \( E = E_1 \cup E_2 \cup E_3 \). We have \( G(\omega) = F(\omega) \) whenever \( \omega \in E_1 \), thus the restriction \( G|_{E_1} \) is scalarly measurable. On the other hand, we also have
\[G(\omega) = \{ x \in F(\omega) : x^*_0(x) = \delta^*(x^*_0, F(\omega)) \} \text{ for every } \omega \in E_2,\]
and Lemma 2.4 can be applied to conclude that \( G|_{E_2} \) is scalarly measurable. In order to finish the proof it only remains to show that \( G|_{E_3} \) is scalarly measurable as well.

By Lemma 5.7, for each \( \omega \in E_3 \) we have
\[(3) \quad G(\omega) = \bigcup_{q \in I(\omega)} \overline{F(\omega) \cap \{ x \in X : x^*_0(x) = q \}}^\text{norm},\]
where \( I(\omega) := \{ q \in \mathbb{Q} : h(\omega) \leq q \leq \delta^*(x^*_0, F(\omega)) \} \). Define \( J(q) := \{ \omega \in E_3 : h(\omega) \leq q \leq \delta^*(x^*_0, F(\omega)) \} \in \Sigma \) for every \( q \in \mathbb{Q} \). Fix \( x^* \in X^* \) and \( a \in \mathbb{R} \), and write \( W := \{ x \in X : x^*(x) > a \} \). Given \( q \in \mathbb{Q} \), Lemma 5.8 ensures that the multi-function \( J(q) \to cwk(X) \) given by \( \omega \mapsto F(\omega) \cap \{ x \in X : x^*_0(x) = q \} \) is scalarly measurable, so the set
\[\{ \omega \in J(q) : F(\omega) \cap \{ x \in X : x^*_0(x) = q \} \cap W \neq \emptyset \}\]
belongs to \( \Sigma \). Since \( W \) is open, equality (3) yields
\[\{ \omega \in E_3 : G(\omega) \cap W \neq \emptyset \} = \bigcup_{q \in I(\omega)} \left( \bigcup_{q \in I(\omega)} F(\omega) \cap \{ x \in X : x^*_0(x) = q \} \right) \cap W \neq \emptyset \]
\[= \bigcup_{q \in \mathbb{Q}} \left( \bigcup_{q \in I(\omega)} F(\omega) \cap \{ x \in X : x^*_0(x) = q \} \right) \cap W \neq \emptyset \] \( \in \Sigma \).

This shows that \( G \) is scalarly measurable. \( \square \)

Let \( \mathcal{M}^n \) be the collection of all finite intersections of closed half-spaces of \( X \).

**Theorem 5.10.** Let \( F : \Omega \to cwk(X) \) be a multi-function. Then \( F \) is scalarly measurable if and only if \( F \) is \( \mathcal{M}^n \)-measurable.

**Proof.** It only remains to check the "only if". We prove the following statement by induction on \( n \in \mathbb{N} \):

(*) For each scalarly measurable multi-function \( G : E \to cwk(X) \), where \( E \in \Sigma \), the set \( \{ \omega \in E : G(\omega) \cap C \neq \emptyset \} \) belongs to \( \Sigma \) whenever \( C \) is the intersection of \( n \) closed half-spaces of \( X \).

The case \( n = 1 \) follows directly from the scalar measurability. Assume \( n > 1 \) and the induction hypothesis. Fix a scalarly measurable multi-function \( G : E \to cwk(X) \), where \( E \in \Sigma \). Take \( C := \bigcap_{i=1}^n \{ x \in X : x^*_i(x) \geq a_i \} \), where \( x^*_1, \ldots, x^*_n \in X^* \) and \( a_1, \ldots, a_n \in \mathbb{R} \). Define \( E' := \{ \omega \in E : \delta^*(x^*_n, G(\omega)) \geq a_n \} \in \Sigma \) and consider the multi-function
\[G' : E' \to cwk(X), \quad G'(\omega) := G(\omega) \cap \{ x \in X : x^*_n(x) \geq a_n \},\]
which is scalarly measurable by Lemma 5.9. Define \( C' := \bigcap_{n=1}^{n-1} \{ x \in X : x^*_i(x) \geq a_i \} \).

Now, by induction hypothesis, the set
\[
\{ \omega \in E' : G'(\omega) \cap C' \neq \emptyset \} = \{ \omega \in E : G(\omega) \cap C \neq \emptyset \}
\]
belongs to \( \Sigma \). The proof is over. \( \square \)

The following lemma is a nice tool to get measurable selectors that will also be applied in the next subsection.

**Lemma 5.11.** Let \( T \) be a topological space and \( \mathcal{M} \) a non-empty family of closed subsets of \( T \). Suppose \( \mathcal{M} \) is closed under finite intersections. Let \( g : T \to [0, \infty) \) be a function such that \( g^{-1}([0, a]) \in \mathcal{M} \) for every \( a \geq 0 \). Let \( F : \Omega \to k(T) \) be a \( \mathcal{M} \)-measurable multi-function. Then:

(i) For each \( \omega \in \Omega \), the set
\[
G(\omega) := \{ t \in F(\omega) : g(t) = \inf \{ g(t') : t' \in F(\omega) \} \}
\]
is compact and non-empty.

(ii) The multi-function \( G : \Omega \to k(T) \) is \( \mathcal{M} \)-measurable.

**Proof.** Since \( \mathcal{M} \) is made up of closed sets, \( g \) is lower semicontinuous and (i) follows straightforwardly bearing in mind that each \( F(\omega) \) is compact and non-empty. We divide the proof of (ii) into several steps.

**Step 1.** Fix \( n \in \mathbb{N} \). For each \( m \in \mathbb{N} \) we define \( A_{n,m} := g^{-1}([0, m/2^n]) \in \mathcal{M} \) and \( B_{n,m} := \{ \omega \in \Omega : F(\omega) \cap A_{n,m} \neq \emptyset \} \in \Sigma \). Clearly, \( B_{n,m} \subset B_{n,m+1} \) for every \( m \in \mathbb{N} \) and \( \Omega = \bigcup_{n=1}^{\infty} B_{n,m} \). Define \( C_{n,1} := B_{n,1} \) and \( C_{n,m} := B_{n,m} \setminus B_{n,m-1} \) for every \( m \geq 2 \), so that \( C_{n,1}, C_{n,2}, \ldots \) is a countable partition of \( \Omega \) in \( \Sigma \). Consider the multi-function \( F_n : \Omega \to k(T) \) defined by \( F_n(\omega) := F(\omega) \cap A_{n,m} \) whenever \( \omega \in C_{n,m} \). Then \( F_n \) is \( \mathcal{M} \)-measurable. Indeed, given \( M \in \mathcal{M} \), note that \( A_{n,m} \cap M \in \mathcal{M} \) for every \( m \in \mathbb{N} \) and we have
\[
\{ \omega \in \Omega : F_n(\omega) \cap M \neq \emptyset \} = \bigcup_{m=1}^{\infty} \left( C_{n,m} \cap \{ \omega \in \Omega : F(\omega) \cap (A_{n,m} \cap M) \neq \emptyset \} \right) \in \Sigma
\]
since \( F \) is \( \mathcal{M} \)-measurable.

**Step 2.** Clearly, \( C_{n,m} = C_{n+1,2m-1} \cup C_{n+1,2m} \) and \( A_{n+1,2m-1} \subset A_{n+1,2m} = A_{n,m} \) for every \( n, m \in \mathbb{N} \), by the very definitions. It follows that \( F_{n+1}(\omega) \subset F_n(\omega) \) for every \( \omega \in \Omega \) and every \( n \in \mathbb{N} \). In view of Lemma 5.3, we can define a \( \mathcal{M} \)-measurable multi-function \( H : \Omega \to k(T) \) by \( H(\omega) := \bigcap_{n=1}^{\infty} F_n(\omega) \).

**Step 3.** Given \( \omega \in \Omega \), note that a point \( t \in F(\omega) \) does not belong to \( G(\omega) \) if and only if \( g(t') < m/2^n < g(t) \) for some \( t' \in F(\omega) \) and some \( n, m \in \mathbb{N} \), which is equivalent to saying that \( \omega \in C_{n,m} \) for some \( 1 \leq m' \leq m \) and \( t \not\in A_{n,m} \). It follows that \( G(\omega) = H(\omega) \) for every \( \omega \in \Omega \) and the proof is over. \( \square \)

**Lemma 5.12.** Let \( T \) be a topological space and \( \mathcal{M} \) a non-empty family of closed subsets of \( T \). Suppose \( \mathcal{M} \) is closed under finite interections. Let \( \kappa < \kappa(\mu) \) be a cardinal and write \( \mathcal{M}(\kappa) \) to denote the collection of all interections of at most \( \kappa \) elements of \( \mathcal{M} \). Then a multi-function \( F : \Omega \to k(T) \) is \( \mathcal{M} \)-measurable if and only if it is \( \mathcal{M}(\kappa) \)-measurable.

**Proof.** It only remains to prove the "only if". We will check that \( F \) is \( \mathcal{M}(\kappa) \)-measurable for every cardinal \( \kappa < \kappa(\mu) \) by transfinite induction. Fix such a cardinal and assume that \( F \) is \( \mathcal{M}(\kappa') \)-measurable for every cardinal \( \kappa' < \kappa \). Clearly, the conclusion follows automatically if \( \kappa \) is finite, since \( \mathcal{M} \) is closed under finite interections. So assume that \( \kappa \) is infinite. Take a family \( \{ M_{\alpha} : \alpha < \kappa \} \subset \mathcal{M} \) and define, for each ordinal \( \beta < \kappa \), the set
\[
N_\beta := \bigcap_{\alpha < \beta} M_{\alpha} \in \text{card}(\beta),
\]
so that \( \{ \omega \in \Omega : F(\omega) \cap N_\beta \neq \emptyset \} \in \Sigma \) by induction hypothesis. Given \( \omega \in \Omega \), the net of compact sets \( (F(\omega) \cap N_\beta)_{\beta < \kappa} \) is decreasing and, therefore, we have

\[
\bigcap_{\beta < \kappa} \{ \omega \in \Omega : F(\omega) \cap N_\beta \neq \emptyset \} = \{ \omega \in \Omega : F(\omega) \cap \left( \bigcap_{\beta < \kappa} N_\beta \right) \neq \emptyset \}.
\]

Observe that \( \bigcap_{\beta < \kappa} N_\beta = \bigcap_{\alpha < \kappa} M_\alpha \). Since the intersection of less than \( \kappa(\mu) \) elements of \( \Sigma \) also belongs to \( \Sigma \) and \( \kappa < \kappa(\mu) \), we conclude that

\[
\left\{ \omega \in \Omega : F(\omega) \cap \left( \bigcap_{\alpha < \kappa} M_\alpha \right) \neq \emptyset \right\} \in \Sigma.
\]

This shows that \( F \) is \( \mathcal{M}(\kappa) \)-measurable, as required. \( \square \)

In the next two theorems we apply the previous work to present sufficient conditions on \( X \) to have the \( \mu \)-SMSP. Recall that a norm \( \| \cdot \| \) on \( X \) is said to be strictly convex if \( x = x' \) whenever \( x, x' \in X \) are such that \( \|x\| = \|x'\| = 1 \) and \( \|x + x'\| = 2 \).

**Theorem 5.13.** If \( X \) admits an equivalent strictly convex norm with the property that \( \text{dens}(B_{X^*}, w^*) < \kappa(\mu) \), then \( X \) has the \( \mu \)-SMSP.

*Proof.* Write \( \kappa := \text{dens}(B_{X^*}, w^*) \). Let \( F : \Omega \to \text{cwk}(X) \) be a scalarly measurable multifunction. By Theorem 5.10 and Lemma 5.12, \( F \) is \( \mathcal{M}^{\omega}(\kappa) \)-measurable. Let \( \| \cdot \| \) be an equivalent strictly convex norm with \( \text{dens}(B_{X^*}, w^*) < \kappa(\mu) \) and define \( g : X \to [0, \infty) \) by \( g(x) := \|x\| \). Observe that

\[
g^{-1}([0, a]) = \bigcap_{x^* \in D} \{ x \in X : |x^*(x)| \leq a \} \in \mathcal{M}^{\omega}(\kappa) \quad \text{for every } a \geq 0,
\]

where \( D \subset B_{X^*} \) is any \( w^* \)-dense set with \( \text{card}(D) = \kappa \). Given \( \omega \in \Omega \), the set

\[
G(\omega) := \{ x \in F(\omega) : \|x\| = \inf \{ \|x'\| : x' \in F(\omega) \} \}
\]

contains only one point, say \( f(\omega) \), because \( F(\omega) \in \text{cwk}(X) \) and \( \| \cdot \| \) is \( w \)-lower semicontinuous and strictly convex. Note that the function \( f : \Omega \to X \) is a selector of \( F \). We can now apply Lemma 5.11 (working with the topological space \( (X, w) \)) and considering the family \( \mathcal{M} = \mathcal{M}^{\omega}(\kappa) \) to conclude that \( f^{-1}(C) \in \Sigma \) for every \( C \in \mathcal{M}^{\omega}(\kappa) \), so that \( f \) is scalarly measurable. \( \square \)

A norm \( \| \cdot \| \) on \( X \) is called *locally uniformly rotated* (shortly LUR) if \( \|x_n - x\| \to 0 \) whenever the sequence \( (x_n) \) in \( X \) and \( x \in X \) satisfy \( \|x_n\| \to \|x\| \) and \( \|x_n + x\| \to 2\|x\| \). Clearly, this property implies strict convexity. Many Banach spaces admit an equivalent LUR norm, for instance, the WLD ones, cf. [10, Corollary 1.10, p. 286]. For complete information about renormings in Banach spaces we refer the reader to [10], [19] and [46].

As an application of the previous theorem we obtain:

**Example 5.14.** \( C([0, \omega_1]) \) has the \( \mu \)-SMSP whenever \( \kappa(\mu) > \omega_1 \). Indeed, it is known that \( C([0, \omega_1]) \) admits an equivalent LUR (in particular, strictly convex) norm, because \( [0, \omega_1] \) is a Valdivia compactum, cf. [10, Corollary 1.10, p. 286]. On the other hand, the dual unit ball of any equivalent norm on \( C([0, \omega_1]) \) has \( w^* \)-density character \( \omega_1 \) (bear in mind that this space contains a subspace isomorphic to \( c_0(\omega_1) \)).

A similar argument allows us to give an alternative proof of the previously announced result of Valadier, see [43, Proposition 6].

**Theorem 5.15** (Valadier). If \( X^* \) is \( w^* \)-separable, then \( X \) has the \( \mu \)-SMSP.

*Proof.* Let \( F : \Omega \to \text{cwk}(X) \) be a scalarly measurable multifunction. By Theorem 5.10 and Lemma 5.12, we know that \( F \) is \( \mathcal{M}^{\omega}(N_0) \)-measurable. Fix a countable \( w^* \)-dense set \( \{ x^*_n : n \in \mathbb{N} \} \subset X^* \) and consider the operator

\[
T : X \to \ell^2(\mathbb{N}), \quad T(x) := \left( \frac{x^*_n(x)}{2^n} \right).
\]
Define \( g : X \to [0, \infty) \) by \( g(x) := ||T(x)||_{\ell^2(\mathbb{N})} \). Since \( B_{\ell^2(\mathbb{N})} \) is \( w^* \)-separable, we have \( g^{-1}([0, a]) \in \mathcal{M}^{w^*}(\mathcal{N}_0) \) for every \( a \geq 0 \). Since \( g \) is a \( w \)-lower semicontinuous strictly convex norm on \( X \) (non necessarily equivalent to the original one!), the arguments in the proof of Theorem 5.13 (dealing now with the family of weakly closed sets \( \mathcal{M}^{w^*}(\mathcal{N}_0) \)) ensure that \( F \) admits \( w^* \)-separable multi-function.

It is well known that \( X \) admits an equivalent strictly convex norm whenever \( X^* \) is \( w^* \)-separable, cf. [10, Theorem 2.4, p. 46]. However, the fact that such an \( X \) has the \( \mu \)-SMSP cannot be deduced, in general, from Theorem 5.13 above. Indeed, the Johnson-Lindenstrauss space \( JL_2 \) has \( w^* \)-separable dual but, for every equivalent norm on \( JL_2 \), the corresponding dual unit ball is not \( w^* \)-separable, see [26, Example 1].

The technique used in the proof of Theorem 2.6 can be used to prove Theorem 5.16 below: the particular case of Banach spaces having \( w^* \)-separable dual was first proved by Valadier in [43, Proposition 7].

**Theorem 5.16.** Suppose \( X \) has the \( \mu \)-SMSP. Let \( F : \Omega \to cwk(X) \) be a scalarly measurable multi-function. Then there is a collection \( \{f_\alpha\}_{\alpha < \text{dens}(X^*, w^*)} \) of scalarly measurable selectors of \( F \) such that

\[
F(\omega) = \{f_\alpha(\omega) : \alpha < \text{dens}(X^*, w^*)\} \quad \text{for every } \omega \in \Omega.
\]

5.2. **Borel measurable selectors.** In this subsection we exploit Lemma 5.11 in order to find nice selectors for multi-functions with stronger measurability properties. It is convenient to recall first some facts concerning measurability in Banach spaces.

Let \( \mathcal{M}^{cc} \) (resp. \( \mathcal{M}^{cc} \)) be the collection of all norm closed (resp. convex closed) subsets of \( X \). Write \( \sigma(\mathcal{M}^{cc}) \) to denote the smallest \( \sigma \)-algebra on \( X \) containing \( \mathcal{M}^{cc} \). In general, we have

\[
\text{Baire}(X, w) \subset \sigma(\mathcal{M}^{cc}) \subset \text{Borel}(X, w) \subset \text{Borel}(X, \text{norm}).
\]

All these \( \sigma \)-algebras coincide for separable \( X \) but some inclusions may be strict beyond the separable case. Talagrand [41] showed that \( \text{Borel}(\ell_\infty(\mathbb{N}), w) \neq \text{Borel}(\ell_\infty(\mathbb{N}), \text{norm}) \) and Edgar [14] proved that the equality \( \text{Borel}(X, w) = \text{Borel}(X, \text{norm}) \) holds whenever \( X \) admits an equivalent Kadec norm (i.e. a norm for which the weak and norm topologies coincide on the unit sphere; clearly, every LUR norm is Kadec). A result of Raja [36, Theorem 1.2] states that \( X \) admits an equivalent LUR norm if and only if every norm open set \( U \subset X \) can be written as \( U = \bigcup_{n \in \mathbb{N}} (C_n \setminus D_n) \), where \( C_n, D_n \in \mathcal{M}^{cc} \) for every \( n \in \mathbb{N} \); in this case, we have \( \sigma(\mathcal{M}^{cc}) = \text{Borel}(X, \text{norm}) \). On the other hand, it is known that \( \text{Baire}(X, w) \neq \sigma(\mathcal{M}^{cc}) \) whenever \( X^* \) is not \( w^* \)-separable, cf. [22, Theorem 1.5.3], but also for \( \ell_\infty(\mathbb{N}) \) and the Johnson-Lindenstrauss spaces [26], see [37, Theorem 2.3].

**Theorem 5.17.** Suppose \( X \) admits an equivalent strictly convex norm. Then every \( \mathcal{M}^{cc} \)-measurable multi-function \( F : \Omega \to cwk(X) \) admits a \( \sigma(\mathcal{M}^{cc}) \)-measurable selector.

**Proof.** Fix an equivalent strictly convex norm \( \|\cdot\| \) on \( X \). Given \( \omega \in \Omega \), the set

\[
G(\omega) := \{x \in F(\omega) : \|x\| = \inf\{\|x'\| : x' \in F(\omega)\}\}
\]

contains only one point \( f(\omega) \) because \( F(\omega) \in cwk(X) \) and \( \|\cdot\| \) is \( w \)-lower semicontinuous and strictly convex. The function \( f : \Omega \to X \) is a selector of \( F \). Obviously, the mapping \( g : X \to [0, \infty) \) given by \( g(x) := \|x\| \) satisfies \( g^{-1}([0, a]) \in \mathcal{M}^{cc} \) for every \( a \geq 0 \). We can apply Lemma 5.11 (working with the topological space \( (X, w) \)) to conclude that \( f \) is \( \sigma(\mathcal{M}^{cc}) \)-measurable.

\[
F(\omega) = \{f_\alpha(\omega) : \alpha < \text{dens}(X)\} \quad \text{for every } \omega \in \Omega.
\]

In fact, under the same assumption we can say more:

**Theorem 5.18.** Suppose \( X \) admits an equivalent strictly convex norm. Then every \( \mathcal{M}^{cc} \)-measurable multi-function \( F : \Omega \to cwk(X) \) admits a collection \( \{f_\alpha\}_{\alpha < \text{dens}(X)} \) of \( \sigma(\mathcal{M}^{cc}) \)-measurable selectors such that

\[
F(\omega) = \{f_\alpha(\omega) : \alpha < \text{dens}(X)\} \quad \text{for every } \omega \in \Omega.
\]
Theorem 5.20. Suppose \( X \) admits an equivalent strictly convex norm \( \| \cdot \| \) on \( X \). Fix \( \alpha < \kappa \). Since the multi-function \( F_\alpha : \Omega \to \text{cwk}(X) \) given by \( F_\alpha(\omega) := -x_\alpha + F(\omega) \) is \( M^{cc} \)-measurable, a glance at the proof of Theorem 5.17 reveals that \( F_\alpha \) admits a \( \sigma(M^{cc}) \)-measurable selector \( g_\alpha : \Omega \to X \) with the property that

\[
\| g_\alpha(\omega) \| = \inf \{ \| x - x_\alpha \| : x \in F(\omega) \}
\]
for every \( \omega \in \Omega \).

Let us consider the \( \sigma(M^{cc}) \)-measurable selector \( f_\alpha : \Omega \to F \) defined by the formula

\[
f_\alpha(\omega) := g_\alpha(\omega) + x_\alpha.
\]
We claim that the collection \( \{ f_\alpha \}_{\alpha < \kappa} \) fulfills the required property. Indeed, fix \( \omega \in \Omega \) and \( x \in F(\omega) \). Given \( \varepsilon > 0 \), there is \( \alpha < \kappa \) such that \( \| x - x_\alpha \| \leq \varepsilon \), hence (4) yields

\[
\| f_\alpha(\omega) - x \| \leq \| g_\alpha(\omega) \| + \| x - x_\alpha \| \leq 2\| x - x_\alpha \| \leq 2\varepsilon.
\]
As \( x \in F(\omega) \) and \( \varepsilon > 0 \) are arbitrary, we get \( F(\omega) = \{ f_\alpha(\omega) : \alpha < \kappa \} \).

As we have mentioned at the beginning of the subsection, if \( X \) admits an equivalent LUR norm then \( \sigma(M^{cc}) = \text{Borel}(X, \text{norm}) \). Bearing in mind that every LUR norm is strictly convex, from Theorem 5.18 we deduce the following corollary.

Corollary 5.19. Suppose \( X \) admits an equivalent LUR norm. Let \( F : \Omega \to \text{cwk}(X) \) be a \( M^{cc} \)-measurable multi-function. Then \( F \) admits a collection of \( \text{Borel}(X, \text{norm}) \)-measurable selectors \( \{ f_\alpha \}_{\alpha < \text{dens}(X)} \) such that

\[
F(\omega) = \{ f_\alpha(\omega) : \alpha < \text{dens}(X) \}
\]
for every \( \omega \in \Omega \).

We stress that the previous corollary improves a result of Leese [30, Theorem 2], who proved the existence of \( \text{Borel}(X, \text{norm}) \)-measurable selectors for \( M^{cc} \)-measurable multi-functions when \( X \) admits an equivalent uniformly rotund norm.

Similar arguments to those of Theorems 5.17 and 5.18, now dealing with the norm topology of \( X \), allow us to deduce the following result.

Theorem 5.20. Suppose \( X \) admits an equivalent strictly convex norm. Let \( F : \Omega \to \text{ck}(X) \) be a \( M^{cc} \)-measurable multi-function. Then \( F \) admits a collection \( \{ f_\alpha \}_{\alpha < \text{dens}(X)} \) of \( \text{Borel}(X, \text{norm}) \)-measurable selectors such that

\[
F(\omega) = \{ f_\alpha(\omega) : \alpha < \text{dens}(X) \}
\]
for every \( \omega \in \Omega \).

Under such assumptions, the existence of at least one \( \text{Borel}(X, \text{norm}) \)-measurable selector was first proved by Leese [30, Theorem 1].

To the best of our knowledge, the question below remains unanswered in full generality:

**OPEN PROBLEM.** Does every Banach space have the \( \mu \)-SMSP for any \( \mu \)?

**References**


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