The fundamental theorem of asset pricing under small transaction costs

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Basic setting of Mathematical Finance:

$(S_t)_{0 \leq t \leq T}$ stochastic process modelling the price of a risky asset ("stock").

$B_t \equiv 1$, for $0 \leq t \leq T$ : riskfree "bond".

Typical Question (Bachelier 1900, Black-Merton-Scholes 1973):

Pricing and Hedging of options like

$$C_T = (S_T - K)_+$$
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**Typical Question (Bachelier 1900, Black-Merton-Scholes 1973):**

Pricing and Hedging of options like

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Basic Result:

**Fundamental Theorem of Asset Pricing**

Under suitable assumptions we have:

\((S_t)_{0 \leq t \leq T}\) does not allow for an arbitrage iff there is an equivalent martingale measure \(Q \sim \mathbb{P}\) for \(S\).

Ross '76
Harrison–Kreps '79
Harrison–Pliska '81
Kreps '81

Delbaen-S. '94,'98.
Corollary (sometimes called "second fundamental theorem of asset pricing"):

If there is a unique equivalent martingale measure $Q$ for the process $(S_t)_{0 \leq t \leq T}$ then the option $C_T$ above (in fact, any $\mathcal{F}_T$-measurable, $Q$-integrable function) can be represented as

$$C_T = \mathbb{E}_Q[C_T] + \int_0^T H_t \, dS_t,$$

for suitable "hedging strategy" $(H_t)_{0 \leq t \leq T}$.

Application:

- $S_t = S_0 + \sigma W_t, \quad 0 \leq t \leq T$ (Bachelier 1900).
- $S_t = S_0 e^{\sigma W_t + \mu t}, \quad 0 \leq t \leq T$ (Samuelson 1965).

Mathematical tool:

"Martingale representation theorem" (K. Itô).
Theorem

([Delbaen, S. 1994]): Let \((S_t)_{0 \leq t \leq T}\) be a locally bounded process which fails to be a semi-martingale (e.g. fractional Brownian motion with \(H \neq \frac{1}{2}\)). Then \((S_t)_{0 \leq t \leq T}\) allows for a **free lunch with vanishing risk** by simple integrands.

More precisely: there is \(\alpha > 0\) such that, for \(\varepsilon > 0\) and \(M > 0\), there is a simple integrand \(H = \sum_{i=1}^{N} H_i \mathbb{1}_{[t_{i-1}, t_i]}\) such that

\[
(H \cdot S)_T \geq -\varepsilon, \quad \text{a.s}
\]

and

\[
\mathbb{P}[(H \cdot S)_T \geq M] \geq \alpha.
\]

Compare also Rogers ’97, Cheridito ’03, Sottinen-Valkeila ’03.
**But:** If we introduce transaction costs of $\varepsilon > 0$, the arbitrage possibilities disappear in a wide class of models, containing (exponential) fractional Brownian motion. [Guasoni, Rasonyi, Schachermayer '08]

Formal setting: Let $(S_t)_{0 \leq t \leq T}$ be an $\mathbb{R}_+$-valued stochastic process and $\varepsilon > 0$.

Assume that $S$ is continuous.

- ask price: $S_t(1 + \varepsilon)$
- bid price: $S_t/(1 + \varepsilon)$

Davis-Norman ´90, Jouini-Kallal '95, Cvitanic-Karatzas '96, Kabanov, Stricker, Touzi, Rasonyi,....
Trading strategies:

Predictable processes \((\vartheta_t)_{0 \leq t \leq T}\) of \textit{finite variation} and satisfying \(\vartheta_0 = \vartheta_T = 0\): "trading strategy".

Value process:

\[
V_t^\varepsilon(\vartheta) = \int_0^t \vartheta_u dS_u - \varepsilon \int_0^t S_u \, d\text{Var}_u(\vartheta)
\]

well defined a.s. as a pathwise Stieltjes integral.

Campi, S. 2006 show that this forms indeed the \textit{natural} class of integrands.
Admissibility of value processes:

Two versions of admissibility:

**Version A** (Harrison-Pliska ’81,...Delbaen-S. ´94, ´98)

\[ V_t^\varepsilon (\vartheta) \geq -M \quad \text{a.s.}, \]

for each \( 0 \leq t \leq T \) and some \( M > 0 \).

**Version B** (Merton ´73,...,Sin ´96, Yan ´98, Jarrow-Protter-Shimbo ´08)

\[ V_t^\varepsilon (\vartheta) \geq -M(1 + S_t) \quad \text{a.s.}, \]

for each \( 0 \leq t \leq T \) and some \( M > 0 \).
**Definition**

The stochastic process \((S_t)_{0\leq t\leq T}\) allows for an arbitrage under \(\varepsilon\) transaction costs (for \(\varepsilon > 0\) fixed) if there is an admissible value process \((V_t^\varepsilon(\vartheta))_{0\leq t\leq T}\) s.t.

\[
P[V_T^\varepsilon(\vartheta) \geq 0] = 1,\]

\[
P[V_T^\varepsilon(\vartheta) > 0] > 0.
\]

**Remark**

Depending on the choice of the concept of admissibility there are presently two versions of the concept of (no) arbitrage.
The analogue to the concept of equivalent (local) martingale measures:

**Definition (Jouini-Kallal ’95,...)**

An $\varepsilon$-consistent price system for the given process $(S_t)_{0 \leq t \leq T}$ is a pair $((\tilde{S}_t)_{0 \leq t \leq T}, Q)$ s.t. $\tilde{S}$ is an $\mathbb{R}_+$-valued stochastic process satisfying

(i) $\frac{1}{1+\varepsilon} \leq \frac{\tilde{S}_t}{S_t} \leq 1 + \varepsilon$, a.s. for all $0 \leq t \leq T$,

(ii) $Q \sim \mathbb{P}$,

(iii) Version A: $(\tilde{S}_t)_{0 \leq t \leq T}$ is a local martingale under $Q$.

Version B: $(\tilde{S}_t)_{0 \leq t \leq T}$ is a true martingale under $Q$. 
Theorem

(Guasoni-Rasonyi-S. 2008): Let \((S_t)_{0 \leq t \leq T}\) be an \(\mathbb{R}_+\)-valued continuous stochastic process adapted to \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\).

T.F.A.E.

(i) For each \(\varepsilon > 0\), \(S\) does not allow for an arbitrage under \(\varepsilon\) transaction costs.

(ii) For each \(\varepsilon > 0\), \(S\) admits an \(\varepsilon\)-consistent price system.

Remark

Remark: The theorem holds true in Version A as well as in Version B.
Proof of Theorem: (sketch of ideas)

(ii) ⇒ (i) easy (as usual):
Make the easy observation that it is better to trade on \((\tilde{S}_t)_{0 \leq t \leq T}\), without transaction costs, than to trade on \((S_t)_{0 \leq t \leq T}\) with \(\varepsilon\) transaction costs because of

\[
S_t/(1 + \varepsilon) \leq \tilde{S}_t \leq S_t(1 + \varepsilon).
\]
(i) $\Rightarrow$ (ii) is the non-trivial part of the theorem. Assuming NA under $\varepsilon$ transaction costs, let us construct $\tilde{S}$ and $Q$.

Define the stopping time $\rho_0$ by

$$\rho_0 = \inf\{ t : \frac{S_t}{S_0} \text{ equals } 1 + \varepsilon \text{ or } \frac{1}{1 + \varepsilon} \} \wedge T$$
The subsequent analysis reduces to the following cases:

Case 1: $\mathbb{P}[A_+] > 0$, $\mathbb{P}[A_-] > 0$, $\mathbb{P}[A_0] > 0$.

Case 2: $\mathbb{P}[A_+] > 0$, $\mathbb{P}[A_-] > 0$, $\mathbb{P}[A_0] = 0$. 
Define the desired measure $Q \sim \mathbb{P}$ on $\mathcal{F}_{\rho_0}$ in such a way that $Q[A_+] = \frac{1}{2+\varepsilon}$ and $Q[A_-] = \frac{1+\varepsilon}{2+\varepsilon}$.

Define $(\tilde{S}_t)_{0 \leq t \leq \rho_0}$ by letting

$$\tilde{S}_t = \mathbb{E}_Q[S_{\rho_0} | \mathcal{F}_t], \quad 0 \leq t \leq \rho_0.$$ 

and observe that

$$\tilde{S}_0 = Q[A_+]S_0(1 + \varepsilon) + Q[A_-]S_0/(1 + \varepsilon) = S_0$$

and that $(\tilde{S}_t)_{0 \leq t \leq \rho_0}$ remains in the "$\varepsilon$-corridor"
The inequality \( \frac{1}{1+\varepsilon} \leq \frac{\tilde{S}_t}{S_t} \leq 1 + \varepsilon \) then is satisfied for \( 0 \leq t \leq \rho_0 \), and \((\tilde{S}_t)_{0\leq t\leq \rho_0}\) is a Q-martingale.

Idea of continuation of construction:
As \( \tilde{S}_{\rho_0} = S_{\rho_0} \) we may iterate the procedure by letting

\[
\rho_1 = \inf\{ t \geq \rho_0 : \frac{S_t}{S_{\rho_0}} \text{ is either } 1 + \varepsilon \text{ or } \frac{1}{1 + \varepsilon} \} \wedge T
\]

etc, etc.
Let us now turn to
Case 1: \((\mathbb{P}[A_0] > 0, \mathbb{P}[A_+] > 0, \mathbb{P}[A_-] > 0)\).
Assume (essentially w.l.g.) that \(S_T = S_0\) on \(A_0\). We now have one
degree of freedom in the construction of \(Q\).
To define $Q$, choose $0 < \lambda < 1$, and let

\[
Q[A_0] = \lambda, \quad Q[A_+] = (1 - \lambda) \frac{1}{2 + \varepsilon}, \quad Q[A_-] = (1 - \lambda) \frac{1 + \varepsilon}{2 + \varepsilon}.
\]

\[
\Rightarrow \tilde{S}_0 = \mathbb{E}_Q[S_{\rho_0}] = S_0
\]
Remark

If $S$ has "conditional full support" in $C([0, T], \mathbb{R}_+)$ w.r. to $\| \cdot \|_\infty$, then we are always in case 1 of the above construction and therefore have in every step one (conditional) degree of freedom $0 < \lambda < 1$. 
This allows for the construction of "many" \( \varepsilon \)-consistent price systems \((\tilde{S}, Q)\). These may e.g. be used to give easy "dual proofs" of the so-called "face lifting" theorems (Soner, Shreve, Cvitanic '95, Levental, Skorohod '97).
Face Lifting Theorem (Levental-Skorohod ’96, Soner-Shreve-Cvitanic ’95,...,Guasoni-Rasonyi-S. ’08):

Suppose that \( S = (S_t)_{0 \leq t \leq T} \) has conditional full support in \( C_+[0, T] \) and suppose \( \varepsilon > 0 \) as transaction costs. Then the cheapest way to superreplicate an option \( C_T = (S_T - K)_+ \), i.e., the smallest constant such that there is \( H \) satisfying

\[
C_T \leq \text{constant} + \int_0^T H_t \, dS_t - \varepsilon \int_0^T S_t \, d\text{Var}_t(\mathcal{Q})
\]

is to take

\[
\text{constant} = S_0, \quad H_t \equiv 1.
\]

Summing up:

In the presence of (even very small) transaction costs, the paradigm of replication/super-replication cannot provide any non-trivial information for the problem of pricing and hedging derivatives.
Utility maximisation (portfolio optimisation) does make good sense also in the presence of transaction costs:

\[ u(x) = \sup_{\vartheta} \mathbb{E}[U(x + \int_0^T \vartheta_t \, dS_t - \varepsilon \int_0^T S_t \, d\text{Var}_t(\vartheta))], \quad x \in \mathbb{R}_+. \]

where \( U(x) \) is a fixed concave, increasing function (e.g. \( U(x) = \log(x) \)).

This problem still makes sense for "random endowment" \( X_T \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}) \) (e.g. \( X_T = C_T \)):

\[ u(X_T) = \sup_{\vartheta} \mathbb{E}[U(X_T + \int_0^T \vartheta_t \, dS_t - \varepsilon \int_0^T S_t \, d\text{Var}_t(\vartheta))]. \]

Utility indifference pricing (de Finetti: "certainty equivalent"):

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$$u(x) = u(X_T)$$
Let $\hat{x}$ and $\hat{X}_T$ be the optimizing strategies corresponding to $x$ and $X_T$; the difference $\hat{X}_T - \hat{x}$ may be interpreted as a hedging strategy for $X_T$.

Research programm:
derive an asymptotic expansion for $\varepsilon \to 0$ and $H \to \frac{1}{2}$ how the option prices and hedging strategies deviate from the classical Black-Scholes price (compare Fouque-Papanicolaou-Sircar, Janecek-Shreve, Kramkov-Sirbu etc.).
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