Metric embeddings: embeddings of discrete metric spaces into Banach spaces

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One of the reasons for usefulness of this idea consists in the fact that for “well-structured” spaces one can apply well-developed tools which are generally not available for discrete metric spaces.
Applications of Metric Embeddings

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- In Geometric Group Theory metric embeddings are used to find an important classification of infinite finitely generated groups.

- In Computer Science metric embeddings are used for construction of polynomial Approximation Algorithms for problems for which (exact) polynomial algorithms are known to be hard to find (in the sense that they can exist only if $P = NP$ in the famous open "P vs NP" problem).

- Reminder: A polynomial algorithm is an algorithm for which there exists a polynomial $Q$ such that the number of steps for any instance of the problem can be estimated from above by $Q$ (the size of the data for that instance).

- In Topology metric embeddings are used to prove special cases of the Novikov and Baum-Connes conjectures (also metric embeddings indicated the direction in which counterexamples to some strengthened forms of the Baum-Connes conjecture were found).
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Let $G$ be a group and $S$ be a finite subset of $G$. We say that $S$ is a generating set of $G$, if each $g \in G$ can be written as a finite product of elements from $S$ (elements from $S$ can be repeated in this product arbitrarily many times). We introduce the distance between elements $g, h \in G$ as the length of the shortest representation of $g^{-1}h$ in terms of elements of $S$. It is easy to check that this is a metric, it is called the word metric.
On applications in group theory

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- Applications of metric embeddings in geometric group theory are based on the fact that in some important properties of groups can be characterized in terms of existence of sufficiently good embeddings into certain Banach spaces.
The sparsest cut problem: We are given a connected graph $G = (V, E)$, with a positive weight (called a capacity) $c(e)$ associated to each edge $e \in E$, and a nonnegative number (called a demand) $D(u, v)$ associated to each (unordered) pair of vertices $u, v \in V$.

The sparsity of the cut $(S, \overline{S})$ is defined as

$$\frac{\sum_{u \in S, v \in \overline{S}, uv \in E} c(uv)}{\sum_{u \in S, v \in \overline{S}} D(u, v)},$$

that is, the sparsity is the ratio between the capacities and the demands which "cross" the cut.

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The sparsest cut problem is known to be computationally hard (in the sense that polynomial algorithms for it can exist only if $P$ is equivalent to $NP$ in the “$P$ vs $NP$” problem), for this reason the following version of the sparsest cut problem is also of interest: find an approximation algorithm for the sparsest cut problem.
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An \( \alpha \)-approximation algorithm for a combinatorial optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor \( \alpha \) of the value of an optimal solution.
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An $\alpha$-approximation algorithm for a combinatorial optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor $\alpha$ of the value of an optimal solution.

In many cases we have to allow $\alpha$ to be a slowly increasing function of one of the parameters of the problem.
An approximation algorithm for the sparsest cut problem can be found on the following lines.

Cut semimetrics (sometimes called elementary cut metrics) let $S$ be a subset of a set $A$, $\bar{S}$ be the complement of $S$. The pair $(S, \bar{S})$ is called a cut in $A$ and $S$, $\bar{S}$ are called parts of this cut. The cut semimetric on $A$ corresponding to the cut $(S, \bar{S})$ is defined by

$$d_S(u, v) = \begin{cases} 0 & \text{if } u \text{ and } v \text{ are in the same part} \\ 1 & \text{if } u \text{ and } v \text{ are in different parts} \end{cases}$$

It is easy to see that the quantity which is of interest for the sparsest cut problem can be rewritten as

$$\sum_{uv \in E} c(uv) d_S(u, v)$$

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The next step in the approximation algorithm is the observation that if we replace the minimum over cut metrics in the sparsest cut problem by the minimum over all metrics, we get a problem for which polynomial algorithms are known (it becomes a Linear Programming problem).

It turns out (and not difficult to prove) that the quotient between this ratio and the optimal ratio can be estimated in terms of the distortion of the optimal embedding of the obtained metric space into Banach space $\ell_1$. I recall the definition of $\ell_1$ (distortion will be introduced later):

$$\ell_1 = \left\{ \left\{ x_i \right\}_{i=1}^{\infty} : x_i \in \mathbb{R}, \left\| \left\{ x_i \right\}_{i=1}^{\infty} \right\| = \sum_{i=1}^{\infty} |x_i| < \infty \right\}.$$
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A manifold $M$ is called aspherical if all of its homotopy groups $\pi_n(M)$ are trivial (contain only the identity).

**Borel conjecture:** Let $X$ and $Y$ be compact aspherical manifolds. If $X$ and $Y$ are homotopy equivalent, then $X$ and $Y$ are homeomorphic.

Reminder: Topological spaces $X$ and $Y$ are called homotopy equivalent if there exist maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g$ is homotopic to the identity of $Y$ and $g \circ f$ is homotopic to the identity on $X$. 

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Metric embeddings
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What do we need to know about embeddings?

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- Let \((X, d_X)\) be a metric space \(Y\) be a Banach space. The *distortion* \(c_Y(X)\) of embeddings of \(X\) into \(Y\) is defined as the infimum of \(C \geq 1\) for which there is a map \(f : X \to Y\) satisfying

\[
\forall u, v \in X \quad d_X(u, v) \leq ||f(u) - f(v)||_Y \leq Cd_X(u, v).
\]

We let \(c_Y(X) = \infty\) if there are no embeddings satisfying (2).

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Metric embeddings
The most important for applications to Approximation Algorithms are embeddings of finite sets into the Hilbert space $\ell_2$ and the space $\ell_1$. The corresponding distortions are denoted by $c_2(X)$ and $c_1(X)$, respectively (instead of $c_{\ell_2}(X)$ and $c_{\ell_1}(X)$).
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The general problem of interest for the area of Approximation Algorithms is to estimate distortions of different classes of spaces. Let mention some important facts.

Bourgain (1985) proved that there exists constant $C < \infty$ such that for any $n$-element metric space $X$ we have $c_2(X) \leq C \ln n$. (Well-known results of Banach space theory imply that $c_1(X) \leq c_2(X)$ for any finite metric space $X$.)

Linial-London-Rabinovich (1995) proved that (up to the value of the constant $C$) Bourgain’s result is optimal (see the next slide).
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For a graph $G$ with vertex set $V$ and a subset $F \subset V$ by $\partial F$ we denote the set of edges connecting $F$ and $V \setminus F$. The expanding constant (a.k.a. Cheeger constant) of $G$ is

$$h(G) = \inf \left\{ \frac{|\partial F|}{\min\{|F|, |V \setminus F|\}} : F \subset V, \ 0 < |F| < +\infty \right\}$$

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A sequence $\{G_n\}$ of graphs is called a **family of expanders** if all of $G_n$ are finite, connected, $k$-regular for some $k \in \mathbb{N}$ (this means that each vertex is incident with exactly $k$ edges), their expanding constants $h(G_n)$ are bounded away from 0 (that is, there exists $\varepsilon > 0$ such that $h(G_n) \geq \varepsilon$ for all $n$), and their orders (numbers of vertices) tend to $\infty$ as $n \to \infty$. 

We consider $G_n$ as metric spaces, whose elements are vertices of the graph $G_n$ and the metric is the number of edges in the shortest walk between the vertices. Linial-London-Rabinovich (1995) proved that $c_1(G_n) \geq c(\varepsilon, k) \ln(|V(G_n)|)$.
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It is far from being clear that such graphs exist. However, it was observed simultaneously with introducing the definition (Kolmogorov–Bardzin’ (1967) and Pinsker (1973)) that suitably defined families of random graphs are families of expanders.

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Reminders:

The girth of a graph $G$ is the number of edges in a shortest cycle in $G$ (we define girth only for graphs containing some cycles). Notation: $g(G)$.

The existence of $k$-regular ($k \geq 3$) families of graphs with indefinitely growing girths is also far from being obvious. The earliest constructions I am aware of were suggested by Erdős-Sachs (1963).
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Linial-Magen-Naor (2002) at the Symposium on the Theory of Computing (STOC) conference suggested a closely related problem: Can we derive any lower bound on $c_1(G)$ that tends to $\infty$ with $g(G)$?

This problem was repeated in the full version of their paper (GAFA 2002), Matoušek’s collection of open problems on embeddings of metric spaces, and Linial’s ICM talk (2002).

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This problem was repeated in the full version of their paper (GAFA 2002), Matoušek’s collection of open problems on embeddings of metric spaces, and Linial’s ICM talk (2002).
Problems mentioned in the previous slide (in the stated generality) are answered by the following theorem.

Theorem (M.O. 2012):
For each $k \geq 3$ there exists a sequence $\{\tilde{G}_n\}_{n=1}^{\infty}$ of finite $k$-regular graphs with $\lim_{n \to \infty} g(\tilde{G}_n) = \infty$ and $\sup_n c_1(\tilde{G}_n) < \infty$.

The construction used to prove the theorem was inspired by the work of Arzhantseva-Guentner-ˇSpakula (GAFA 2012) on quotients of free groups. I do not plan to describe the connection between constructions as I do not want to introduce algebraic preliminaries.

One of the important (and attractive for some people) features of my proof is that it is completely elementary and requires knowledge of basic Graph Theory only.

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Coarse embeddings

- In applications to Group Theory and Topology the following class of embeddings is important.

Let $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ be two non-decreasing functions (important: $\rho_2$ has finite values), and let $F : (X, d_X) \to (Y, d_Y)$ be a mapping between two metric spaces such that

$$\forall u, v \in X \rho_1(d_X(u, v)) \leq d_Y(F(u), F(v)) \leq \rho_2(d_X(u, v)).$$

The mapping $F$ is called a coarse embedding if $\rho_1$ can be chosen to satisfy $\lim_{t \to \infty} \rho_1(t) = \infty$.
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It is clear that this definition imposes nontrivial restrictions only when we consider embeddings of unbounded metric spaces: If a space $X$ is such that $\sup_{u, v \in X} d_X(u, v) < \infty$, then the map which maps all elements of $X$ to the same element of $Y$ is a coarse embedding.
Examples of coarse embeddings

Example 1. The mapping $F : \mathbb{R} \to \mathbb{Z}$ given by $F(x) = \lfloor x \rfloor$ is a coarse embedding.
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- **Example 2.** The vertex set $V$ of an infinite dyadic tree $T$ with its graph distance can be coarsely embedded into $\ell_2$ in the following way: we consider a bijection between the set of all edges of $T$ and vectors of an orthonormal basis $\{e_i\}$ in $\ell_2$, and map each vertex from $V$ onto the sum of those vectors from $\{e_i\}$ which correspond to a path from a root $O$ of $T$ to the vertex, $O$ is mapped to 0.

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It turns out that embeddability of a locally finite metric space into a Banach space is finitely determined in the sense explained on the next slide.
Embeddings of locally finite metric spaces are finitely determined

- **Theorem (M.O. 2012):** (1) Let $A$ be a locally finite metric space whose finite subsets admit embeddings into a Banach space $X$ with uniformly bounded distortions. Then $A$ admits an embedding into $X$ with finite distortion.

Before this result was known for many different classes of Banach spaces (such as $L^p(0,1)$), but its validity in the general case was a kind of unexpected.
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The proof for (infinite-dimensional) Banach spaces $X$ which are not isomorphic to $X \oplus \mathbb{R}$ requires an additional step (such spaces $X$ were constructed by Gowers and Maurey (1993)).
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For this result details can be found in my book “Metric embeddings”.

One of the important directions of research in the theory of metric embeddings is: Characterize well-known classes of Banach spaces in terms of their metrics, that is in such a way that characterizations include only distances between vectors of the space and do not include linear combinations or linear functionals.
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An interest to such characterizations is stimulated by the fact that in some of the recent works in the theory of embeddings of metric spaces into Banach spaces (Cheeger, Kleiner, Lee, Naor, 2006–2015) an important role is played by the class of Banach spaces with the RNP.
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In 2009 Bill Johnson suggested the problem: Find a purely metric characterization of the Radon-Nikodým property.
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Introduction to the RNP

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  - Measure-theoretic definition (it gives the name to this property) $X \in \text{RNP} \iff$ The following analogue of the Radon-Nikodým theorem holds for $X$-valued measures.
    - Let $(\Omega, \Sigma, \mu)$ be a positive finite real-valued measure, and $(\Omega, \Sigma, \tau)$ be an $X$-valued measure on the same $\sigma$-algebra which is absolutely continuous with respect to $\mu$ (this means $\mu(A) = 0 \implies \tau(A) = 0$) and satisfies the condition $\tau(A)/\mu(A)$ is a uniformly bounded set of vectors over all $A \in \Sigma$ with $\mu(A) \neq 0$. Then there is an $f \in L_1(\mu, X)$ such that
      $$\forall A \in \Sigma \quad \tau(A) = \int_A f(\omega) d\mu(\omega).$$
Definition in terms of differentiability (going back to Clarkson (1936) and Gelfand (1938)) \( X \in \text{RNP} \iff \text{X-valued Lipschitz functions on } \mathbb{R} \text{ are differentiable almost everywhere.} \)
Further equivalent definitions of the Radon-Nikodým property (RNP)

- Definition in terms of differentiability (going back to Clarkson (1936) and Gelfand (1938)) $X \in \text{RNP} \iff$ $X$-valued Lipschitz functions on $\mathbb{R}$ are differentiable almost everywhere.

- Probabilistic definition (Chatterji (1968)) $X \in \text{RNP} \iff$ Bounded $X$-valued martingales converge.
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- **Definition in terms of differentiability (going back to Clarkson (1936) and Gelfand (1938))** \( X \in \text{RNP} \iff \) \( X \)-valued Lipschitz functions on \( \mathbb{R} \) are differentiable almost everywhere.

- **Probabilistic definition (Chatterji (1968))** \( X \in \text{RNP} \iff \) Bounded \( X \)-valued martingales converge.
  - In more detail: A Banach space \( X \) has the RNP if and only if each \( X \)-valued martingale \( \{f_n\} \) on some probability space \((\Omega, \Sigma, \mu)\), for which \( \{||f_n(\omega)|| : n \in \mathbb{N}, \omega \in \Omega\} \) is a bounded set, converges in \( L_1(\Omega, \Sigma, \mu, X) \).
Further equivalent definitions of the Radon-Nikodým property (RNP)

- Geometric definition. $X \in \text{RNP} \iff$ Each bounded closed convex set in $X$ is dentable in the following sense:
Further equivalent definitions of the Radon-Nikodým property (RNP)

- Geometric definition. $X \in \text{RNP} \iff$ Each bounded closed convex set in $X$ is dentable in the following sense:
  - A bounded closed convex subset $C$ in a Banach space $X$ is called *dentable* if for each $\varepsilon > 0$ there is a continuous linear functional $f$ on $X$ and $\alpha > 0$ such that the set
    \[ \{ y \in C : f(y) \geq \sup \{ f(x) : x \in C \} - \alpha \} \]
    has diameter $< \varepsilon$. 

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Examples

- RNP: Reflexive (for example $L_p$, $1 < p < \infty$), separable dual spaces (for example, $\ell_1$).
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- **RNP**: Reflexive (for example $L^p$, $1 < p < \infty$), separable dual spaces (for example, $\ell_1$).
- **non-RNP**: $c_0$, $L_1(0, 1)$, nonseparable duals of separable Banach spaces.
Our main goal is to present a metric characterization of the RNP in terms of thick families of geodesics.
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We start with a standard definition:

Let \( u \) and \( v \) be two elements in a metric space \( (M, d_M) \). A \( uv \)-geodesic is a distance-preserving map \( g : [0, d_M(u, v)] \to M \) such that \( g(0) = u \) and \( g(d_M(u, v)) = v \) (where \( [0, d_M(u, v)] \) is an interval in \( \mathbb{R} \)).
Thick families of geodesics

A family \( T \) of \( uv \)-geodesics is called \textit{thick} if there is \( \alpha > 0 \) such that for every \( g \in T \) and for every finite collection of points \( r_1, \ldots, r_n \) in the image of \( g \), there is another \( uv \)-geodesic \( \tilde{g} \) satisfying the conditions:

1. The image of \( \tilde{g} \) also contains \( r_1, \ldots, r_n \) (we call these points \textit{control points}).
2. Possibly there are some more common points of \( g \) and \( \tilde{g} \).
3. We can find a sequence \( 0 < s_1 < q_1 < s_2 < q_2 < \cdots < s_m < q_m < s_{m+1} \) such that \( g(q_i) = \tilde{g}(q_i) \) (\( i = 1, \ldots, m \)) are common points containing \( r_1, \ldots, r_n \), and the images \( g(s_i) \) and \( \tilde{g}(s_i) \) are distinct and the sum of deviations over them is nontrivially large in the sense that
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   \sum_{i=1}^{m+1} d_M(g(s_i), \tilde{g}(s_i)) \geq \alpha.
   \]
4. Furthermore, each geodesic which on some intervals between the points \( 0 = q_0 < q_1 < q_2 < \cdots < q_m = d_M(u, v) \) coincides with \( g \) and on others with \( \tilde{g} \) is also in \( T \).
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  - (4) Furthermore, each geodesic which on some intervals between the points $0 = q_0 < q_1 < q_2 < \cdots < q_m = d_M(u, v)$ coincides with $g$ and on others with $\tilde{g}$ is also in $T$. 

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Metric embeddings
Examples: Numerous examples of thick families of geodesics can be obtained by considering a metric space consisting of one geodesic joining two points and then adding more and more geodesics in infinitely many steps.
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Theorem 1 (M.O. (2014)). A Banach space $X$ does not have the RNP if and only if there exists a metric space $M_X$ containing a thick family $T_X$ of geodesics which admits an embedding into $X$ with finite distortion.

At the same time, it turns out that the metric space $M_X$ in Theorem 1 cannot be chosen independently of $X$ because the following result holds.

Theorem 2 (M.O. (2014)). For each metric space $M$ containing a thick family of geodesics there exists a Banach space $X$ which does not have the RNP and does not admit an embedding of $M$ with finite distortion.
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