Metric embeddings: embeddings of discrete metric spaces into Banach spaces

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Introduction

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- One of the reasons for usefulness of this idea consists in the fact that for "well-structured" spaces one can apply well-developed tools which are generally not available for discrete metric spaces.

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  - In Topology metric embeddings are used to prove special cases of the Novikov and Baum-Connes conjectures (also metric embeddings indicated the direction in which counterexamples to some strengthened forms of the Baum-Connes conjecture were found).

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- Applications of metric embeddings in geometric group theory are based on the fact that in some important properties of groups can be characterized in terms of existence of sufficiently good embeddings into certain Banach spaces.

Computer science: the sparsest cut problem

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- The *sparsity* of the cut $(S, \bar{S})$ is defined as

$$\frac{\sum_{u\in S, v\in\bar{S}, uv\in E} c(uv)}{\sum_{u\in S, v\in\bar{S}} D(u,v)}, \tag{1}$$

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  that is, the sparsity is the ratio between the capacities and the demands which "cross" the cut.
- The *sparsest cut problem*: What is the minimum sparsity among all cuts with nonzero $\sum_{u\in S, v\in \bar{S}} D(u, v)$?

- The sparsest cut problem is known to be computationally hard (in the sense that polynomial algorithms for it can exist only if P is equivalent to NP in the "P vs NP" problem), for this reason the following version of the sparsest cut problem is also of interest: find an approximation algorithm for the sparsest cut problem.

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- In many cases we have to allow α to be a slowly increasing function of one of the parameters of the problem.

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- **Cut semimetrics** (sometimes called *elementary cut metrics*) Let S be a subset of a set A, $\bar{S}$ be the complement of S. The pair $(S, \bar{S})$ is called a *cut* in A and $S, \bar{S}$ are called *parts* of this cut. The *cut semimetric* on A corresponding to the cut $(S, \bar{S})$ is defined by

$$d_S(u,v) = \begin{cases} 0 & \text{if } u \text{ and } v \text{ are in the same part} \\ 1 & \text{if } u \text{ and } v \text{ are in different parts} \end{cases}$$

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- It is easy to see that the quantity which is of interest for the sparsest cut problem can be rewritten as

$$\frac{\sum_{uv\in E} c(uv) d_S(u,v)}{\sum_{u,v\in V} D(u,v) d_S(u,v)} \quad \left(\text{Originally: } \frac{\sum_{u\in S, v\in\bar{S}, uv\in E} c(uv)}{\sum_{u\in S, v\in\bar{S}} D(u,v)} \right)$$

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- It turns out (and not difficult to prove) that the quotient between this ratio and the optimal ratio can be estimated in terms of the distortion of the optimal embedding of the obtained metric space into Banach space ℓ_1. I recall the definition of ℓ_1 (distortion will be introduced later):

$$\ell_1 = \left\{ \{x_i\}_{i=1}^{\infty} : \ x_i \in \mathbb{R}, \ ||\{x_i\}_{i=1}^{\infty}|| = \sum_{i=1}^{\infty} |x_i| < \infty \right\}.$$

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- **Borel conjecture:** Let X and Y be compact aspherical manifolds. If X and Y are homotopy equivalent, then X and Y are homeomorphic.
- **Reminder:** Topological spaces X and Y are called *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity of Y and $g \circ f$ is homotopic to the identity on X.

What do we need to know about embeddings?

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- Let (X, d_X) be a metric space Y be a Banach space. The *distortion* $c_Y(X)$ of embeddings of X into Y is defined as the infimum of $C \geq 1$ for which there is a map $f : X \to Y$ satisfying

$$\forall u, v \in X \quad d_X(u, v) \leq ||f(u) - f(v)||_Y \leq C d_X(u, v). \qquad (2)$$

We let $c_Y(X) = \infty$ if there are no embeddings satisfying (2).

- The most important for applications to Approximation Algorithms are embeddings of finite sets into the Hilbert space ℓ_2 and the space ℓ_1. The corresponding distortions are denoted by $c_2(X)$ and $c_1(X)$, respectively (instead of $c_{\ell_2}(X)$ and $c_{\ell_1}(X)$).

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- Bourgain (1985) proved that there exists constant $C < \infty$ such that for any n-element metric space X we have $c_2(X) \le C \ln n$. (Well-known results of Banach space theory imply that $c_1(X) \le c_2(X)$ for any finite metric space X.)

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- Linial-London-Rabinovich (1995) proved that (up the the value of the constant C) Bourgain's result is optimal (see the next slide).

- For a graph G with vertex set V and a subset $F \subset V$ by ∂F we denote the set of edges connecting F and $V \backslash F$. The *expanding constant* (a.k.a. *Cheeger constant*) of G is

$$h(G) = \inf \left\{ \frac{|\partial F|}{\min\{|F|, |V \backslash F|\}} : \; F \subset V, \; 0 < |F| < +\infty \right\}$$

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- A sequence $\{G_n\}$ of graphs is called a *family of expanders* if all of G_n are finite, connected, *k-regular* for some $k \in \mathbf{N}$ (this means that each vertex is incident with exactly k edges), their expanding constants $h(G_n)$ are bounded away from 0 (that is, there exists $\varepsilon > 0$ such that $h(G_n) \geq \varepsilon$ for all n), and their orders (numbers of vertices) tend to ∞ as $n \to \infty$.

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- We consider G_n as metric spaces, whose elements are vertices of the graph G_n and the metric is the number of edges in the shortest walk between the vertices. Linial-London-Rabinovich (1995) proved that $c_1(G_n) \geq c(\varepsilon, k) \ln(|V(G_n)|)$

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- It was a very important for applications problem: to construct families of expanders deterministically. Eventually many of such families were constructed (starting with Margulis (1973)).

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- **Reminders:** The *girth* of a graph G is the number of edges in a shortest cycle in G (we define girth only for graphs containing some cycles). Notation: $g(G)$.
- The existence of k-regular ($k \geq 3$) families of graphs with indefinitely growing girths is also far from being obvious. The earliest constructions I am aware of were suggested Erdős-Sachs (1963).

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- This problem was repeated in the full version of their paper (GAFA 2002), Matoušek's collection of open problems on embeddings of metric spaces, and Linial's ICM talk (2002).

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- One can find all details and relevant background in my book M. I. Ostrovskii, "Metric embeddings", 2013.

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- In applications to Group Theory and Topology the following class of embeddings is important.
- Let $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ be two non-decreasing functions (important: ρ_2 has finite values), and let $F : (X, d_X) \to (Y, d_Y)$ be a mapping between two metric spaces such that $\forall u, v \in X \;\; \rho_1(d_X(u, v)) \le d_Y(F(u), F(v)) \le \rho_2(d_X(u, v))$. The mapping F is called a *coarse embedding* if ρ_1 can be chosen to satisfy $\lim_{t\to\infty} \rho_1(t) = \infty$.
- It is clear that this definition imposes nontrivial restrictions only when we consider embeddings of unbounded metric spaces: If a space X is such that $\sup_{u,v\in X} d_X(u, v) < \infty$, then the map which maps all elements of X to the same element of Y is a coarse embedding.

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- **Example 2.** The vertex set V of an infinite dyadic tree T with its graph distance can be coarsely embedded into ℓ_2 in the following way: we consider a bijection between the set of all edges of T and vectors of an orthonormal basis $\{e_i\}$ in ℓ_2, and map each vertex from V onto the sum of those vectors from $\{e_i\}$ which correspond to a path from a root O of T to the vertex, O is mapped to 0.

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- In Example 2 the distance between the images is the $\sqrt{}$ of the distance between vertices.

Locally finite metric spaces

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- The main metric space whose embeddability is of interest for applications in Group Theory and Topology is a finitely generated group with its word metric. It is easy to check that each such metric space is locally finite.
- It turns out that embeddability of a locally finite metric space into a Banach space is finitely determined in the sense explained on the next slide.

Embeddings of locally finite metric spaces are finitely determined

- **Theorem (M.O. 2012): (1)** *Let A be a locally finite metric space whose finite subsets admit embeddings into a Banach space X with uniformly bounded distortions. Then A admits an embedding into X with finite distortion.*

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- **(2)** *Let A be a locally finite metric space whose finite subsets admit coarse embeddings into a Banach space X with the same functions ρ_1 and ρ_2 for all subsets. Then A admits a coarse embedding into X.*
- Before this result was known for many different classes of Banach spaces (such as $L_p(0,1)$), but its validity in the general case was a kind of unexpected.

On the proof of the finite determination result

- The proof of the theorem is based on one of the kinds of convergence which exist for bounded sets in an arbitrary separable Banach space and a pasting procedure which was first suggested by Ribe (1984) and later developed by Baudier and Lancien (2008).

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- For this result details can be found in my book "Metric embeddings".

Metric characterizations

- One of the important directions of research in the theory of metric embeddings is: Characterize well-known classes of Banach spaces in terms of their metrics, that is in such a way that characterizations include only distances between vectors of the space and do not include linear combinations or linear functionals.

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- An interest to such characterizations is stimulated by the fact that in some of the recent works in the theory of embeddings of metric spaces into Banach spaces (Cheeger, Kleiner, Lee, Naor, 2006–2015) an important role is played by the class of Banach spaces with the RNP.
- In 2009 Bill Johnson suggested the problem: Find a purely metric characterization of the Radon-Nikodým property.

Introduction to the RNP

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  - Measure-theoretic definition (it gives the name to this property) $X \in \mathrm{RNP} \Leftrightarrow$ The following analogue of the Radon-Nikodým theorem holds for X-valued measures.
    - Let (Ω, Σ, μ) be a positive finite real-valued measure, and (Ω, Σ, τ) be an X-valued measure on the same σ-algebra which is absolutely continuous with respect to μ (this means $\mu(A) = 0 \Rightarrow \tau(A) = 0$) and satisfies the condition $\tau(A)/\mu(A)$ is a uniformly bounded set of vectors over all $A \in \Sigma$ with $\mu(A) \neq 0$. Then there is an $f \in L_1(\mu, X)$ such that

$$\forall A \in \Sigma \quad \tau(A) = \int_A f(\omega) d\mu(\omega).$$

Further equivalent definitions of the Radon-Nikodým property (RNP)

- Definition in terms of differentiability (going back to Clarkson (1936) and Gelfand (1938)) $X \in \mathrm{RNP} \Leftrightarrow X$-valued Lipschitz functions on $\mathbb{R}$ are differentiable almost everywhere.

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- Probabilistic definition (Chatterji (1968)) $X \in \mathrm{RNP} \Leftrightarrow$ Bounded X-valued martingales converge.
  - In more detail: A Banach space X has the RNP if and only if each X-valued martingale $\{f_n\}$ on some probability space (Ω, Σ, μ), for which $\{||f_n(\omega)|| : \ n \in \mathbb{N}, \ \omega \in \Omega\}$ is a bounded set, converges in $L_1(\Omega, \Sigma, \mu, X)$.

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- Geometric definition. $X \in \mathrm{RNP} \Leftrightarrow$ Each bounded closed convex set in X is dentable in the following sense:
  - A bounded closed convex subset C in a Banach space X is called *dentable* if for each $\varepsilon > 0$ there is a continuous linear functional f on X and $\alpha > 0$ such that the set

$$\{y \in C : \ f(y) \geq \sup\{f(x) : \ x \in C\} - \alpha\}$$

    has diameter $< \varepsilon$.

Examples

- RNP: Reflexive (for example L_p, $1 < p < \infty$), separable dual spaces (for example, ℓ_1).

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- non-RNP: c_0, $L_1(0,1)$, nonseparable duals of separable Banach spaces.

Geodesics

- Our main goal is to present a metric characterization of the RNP in terms of thick families of geodesics.

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- Our main goal is to present a metric characterization of the RNP in terms of thick families of geodesics.
- We start with a standard definition:
  Let u and v be two elements in a metric space (M, d_M). A *uv-geodesic* is a distance-preserving map $g : [0, d_M(u, v)] \to M$ such that $g(0) = u$ and $g(d_M(u, v)) = v$ (where $[0, d_M(u, v)]$ is an interval in $\mathbb{R}$).

Thick families of geodesics

- A family T of uv-geodesics is called *thick* if there is $\alpha > 0$ such that for every $g \in T$ and for every finite collection of points $r_1, \dots, r_n$ in the image of g, there is another uv-geodesic $\widetilde{g}$ satisfying the conditions:

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  - (2) Possibly there are some more common points of g and $\widetilde{g}$.
  - (3) We can find a sequence $0 < s_1 < q_1 < s_2 < q_2 < \cdots < s_m < q_m < s_{m+1} < d_M(u, v)$, such that $g(q_i) = \widetilde{g}(q_i)$ $(i = 1, \dots, m)$ are common points containing $r_1, \dots, r_n$, and the images $g(s_i)$ and $\widetilde{g}(s_i)$ are distinct and the sum of deviations over them is nontrivially large in the sense that $\sum_{i=1}^{m+1} d_M(g(s_i), \widetilde{g}(s_i)) \geq \alpha$.

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  - (4) Furthermore, each geodesic which on some intervals between the points $0 = q_0 < q_1 < q_2 < \cdots < q_m = d_M(u, v)$ coincides with g and on others with $\widetilde{g}$ is also in T.

RNP and thick family of geodesics

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- At the same time, it turns out that the metric space M_X in Theorem 1 cannot be chosen independently of X because the following result holds.
- **Theorem 2** (M.O. (2014)). For each metric space M containing a thick family of geodesics there exists a Banach space X which does not have the RNP and does not admit an embedding of M with finite distortion.