The Algebra of symmetric analytic functions of bounded type on the complex $L_\infty$

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(joint work with Pablo Galindo and Andriy Zagorodnyuk)

Let $L_\infty$ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions $x$ on $[0, 1]$ with norm

$$
\|x\|_\infty = \text{ess sup}_{t \in [0, 1]} |x(t)|.
$$

Let $\Xi$ be the set of all measurable bijections of $[0, 1]$ that preserve the measure.

A function $f : L_\infty \to \mathbb{C}$ is called symmetric if for every $x \in L_\infty$ and for every $\sigma \in \Xi$

$$
f(x \circ \sigma) = f(x).
$$
Theorem 1

Polynomials $R_n : L_\infty \to \mathbb{C}$, $R_n(x) = \int_{[0,1]} (x(t))^n \, dt$ for $n \in \mathbb{N}$, form an algebraic basis in the algebra of all symmetric continuous polynomials on $L_\infty$.

Let $H_{bs}(L_\infty)$ be the Fréchet algebra of all entire symmetric functions $f : L_\infty \to \mathbb{C}$ which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets.

Since every $f \in H_{bs}(L_\infty)$ can be described by its Taylor series of continuous symmetric homogeneous polynomials, it follows that $f$ can be uniquely represented as

$$f(x) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1 + 2k_2 + \ldots + nk_n = n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x).$$
Consequently, for every non-trivial continuous homomorphism \( \varphi : H_{bs}(L_{\infty}) \to \mathbb{C} \), taking into account \( \varphi(1) = 1 \), we have
\[
\varphi(f) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} \varphi(R_1)^{k_1} \cdots \varphi(R_n)^{k_n}.
\]

Therefore, \( \varphi \) is completely determined by the sequence of its values on \( R_n : \)
\[
(\varphi(R_1), \varphi(R_2), \ldots).
\]

We denote by \( M_{bs} \) the spectrum of \( H_{bs}(L_{\infty}) \), that is, the set of all continuous homomorphisms \( \varphi : H_{bs}(L_{\infty}) \to \mathbb{C} \).
Proposition 2

For every non-trivial continuous homomorphism $\varphi : H_{bs}(L_\infty) \to \mathbb{C}$, the sequence $\left\{ \sqrt[n]{|\varphi(R_n)|} \right\}_{n=1}^{\infty}$ is bounded.

Theorem 3

For every sequence $\xi = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\sup_{n \in \mathbb{N}} n^{1/n} |\xi_n| < +\infty$, there exists $x_\xi \in L_\infty$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$. 
Let \( \varepsilon_n(t) = \text{sign} \sin 2^n \pi t \)

**Theorem (J.-P. Kahane “Some random series of functions”, 1968)**

The series

\[
\sum_{n=1}^{\infty} \varepsilon_n(t) u_n
\]

is convergent almost everywhere on \([0, 1]\) if and only if the series

\[
\sum_{n=1}^{\infty} u_n^2
\]

is convergent.
Corollary

The series

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n(t)}{n + 1}$$

converges almost everywhere on $[0, 1]$.

Let

$$p_n(t) = \exp \left( \frac{i \pi}{2n} \sum_{k=1}^{\infty} \frac{\varepsilon_k(t)}{k + 1} \right).$$

By the Corollary, $p_n \in L_\infty$.

Note that

$$R_m(p_n) = \prod_{k=1}^{\infty} \cos \left( \frac{\pi m}{2n} \frac{1}{k + 1} \right).$$
Let us define functions \( y_n : [0, 1] \to \mathbb{C} \). For \( t \in \left[ \frac{k-1}{n}, \frac{k}{n} \right) \) let
\[
y_n(t) = \alpha_k p_n(nt - k + 1),
\]
where
\[
\alpha_k = \exp \left( \frac{2\pi ik}{n} \right), \quad (k = 1, 2, \ldots, n).
\]
Note that \( \|y_n\|_\infty = \text{ess sup}_{t \in [0,1]} |y_n(t)| = 1 \).

We have
\[
R_m(y_n) = \int_{[0,1]} (y_n(t))^m \, dt = \sum_{k=1}^{n} \int_{\left[ \frac{k-1}{n}, \frac{k}{n} \right)} (\alpha_k p_n(nt - k + 1))^m \, dt =
\]
\[
= \left( \frac{1}{n} \sum_{k=1}^{n} \alpha_k^m \right) \int_{[0,1]} (p_n(t))^m \, dt.
\]
Since
\[
\frac{1}{n} \sum_{k=1}^{n} \alpha_k^m = \begin{cases} 
1, & \text{if } m \text{ is a multiple of } n, \\
0, & \text{otherwise.}
\end{cases}
\]
it follows that \( R_m(y_n) = 0 \) if \( m \) is not a multiple of \( n \).

If \( m \) is a multiple of \( n \), i.e. \( m = k_0n \) for some \( k_0 \in \mathbb{N} \), we have
\[
R_m(y_n) = \int_{[0,1]} (p_n(t))^m \, dt = \prod_{k=1}^{\infty} \cos \left( \frac{\pi}{2} \cdot \frac{k_0}{k+1} \right).
\]
If \( m \neq n \), then \( k_0 > 1 \) and one of the factors is equal to \( \cos \frac{1}{2} \pi \), and, therefore, \( R_m(y_n) = 0 \). In the \( m = n \) case we have
\[
R_n(y_n) = \prod_{k=1}^{\infty} \cos \left( \frac{\pi}{2} \cdot \frac{1}{k+1} \right).
\]
Let \( M = R_n(y_n) \). Note that \( 0 < M < 1 \).
Let

\[ x_n(t) = \frac{1}{\sqrt[n]{M} y_n(t)}. \]

Note that

\[ R_m(x_n) = \begin{cases} 
1, & \text{if } m = n, \\
0, & \text{otherwise}. 
\end{cases} \]

Let us define \( x_\xi(t) \).

For \( t \in \left[ \frac{2^{n-1} - 1}{2^n - 1}, \frac{2^n - 1}{2^n} \right] \) let

\[ x_\xi(t) = 2^{\frac{n}{\xi_n}} x_n(2^n t - 2^n + 2). \]
Corollary 4

Every $\varphi \in M_{bs}$ is a point-evaluation functional.

Corollary 5

The spectrum $M_{bs}$ can be identified with the set of all sequences $\xi = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\{\sqrt[n]{|\xi_n|}\}_{n=1}^{\infty}$ is bounded.
Let $\nu : L_\infty \to M_{bs}$ be defined by $\nu(x) = (R_1(x), R_2(x), \ldots)$.

Let $\tau_\infty$ be the topology on $L_\infty$, generated by $\| \cdot \|_\infty$.

Let us define an equivalence relation on $L_\infty$ by $x \sim y \iff \nu(x) = \nu(y)$.

Let $\tau$ be the quotient topology on $M_{bs}$:

$$\tau = \{ \nu(V) : V \in \tau_\infty \}.$$ 

**Theorem 6**

$(M_{bs}, +, \tau)$ is an abelian topological group, where “$+$” is an operation of coordinate-wise addition.