Hardy spaces of Dirichlet Series and the Riemann Zeta Function

Kristian Seip

Norwegian University of Science and Technology NTNU

Nonlinear Functional Analysis

Valencia, October 20, 2017

1997 paper with Hedenmalm and Lindqvist

In 1997, I published a paper with Håkan Hedenmalm and Peter Lindqvist in Duke Math. J. which begins with the following words:

“The purpose of this paper is twofold. First we study systems of functions $\varphi(x), \varphi(2x), \varphi(3x), \ldots$, and second we study the Hardy space H^2 of the infinite dimensional polydisk. Building on ideas of Arne Beurling and Harald Bohr, we find that the two topics are intimately connected, the common feature being the use of Dirichlet series.”

Systems of dilated functions—an example

More precisely, we were concerned with the case when φ is an odd 2-periodic function—the prototype being $\varphi(x) = \sin \pi x$. Our interest in systems of the form $\varphi(nx)$ came from a nonlinear eigenvalue problem (for the one-dimensional p-Laplacian):

$$\frac{d}{dx}\left(|u'|^{p-2}u'\right) + \lambda |u|^{p-2}u = 0,$$

where $1 < p < \infty$, studied in depth by Ôtani in 1984. It has a remarkable solution, and all eigenfunctions can be expressed in terms of **dilations of a single odd periodic function**, called $\sin_p(x)$, an inverse of a certain Abelian integral, corresponding to the smallest eigenvalue λ.

Riesz bases of systems of dilated functions

We were led from this special case to a general problem:

Question

When will a system of functions $\varphi(nx)$, $n = 1, 2, 3, \ldots$, φ odd and 2-periodic, be a Riesz basis for $L^2(0,1)$?

When $\varphi(x) = \sqrt{2}\sin\pi x$, we have an orthonormal basis, and the question is somehow how much a general φ can differ from $\sqrt{2}\sin\pi x$ for the Riesz basis property to be "preserved". (A problem well known from systems of exponentials!)

An effective way of expressing analytically such "deviation" comes from work of Beurling (and also Wintner, as we later we discovered).

The Beurling–Wintner transformation

We may write

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sqrt{2} \sin(\pi n x);$$

then

$$\sum_{n=1}^{\infty} b_n \varphi(nx) = \sum_{n} c_n \sqrt{2} \sin(\pi n x), \tag{1}$$

where $c_n = \sum_{d|n} a_n b_{n/d}$ (which is called the Dirichlet convolution of (a_n) and (b_n)). Therefore, the mapping

$$\mathscr{S} \sum_{n=1} a_n \sqrt{2} \sin(\pi n x) := \sum_{n=1}^{\infty} a_n n^{-s}$$

transforms (1) into a bona fide product of $\sum a_n n^{-s}$ and $\sum b_n n^{-s}$. This observation is found in the notes from a seminar held by Beurling in 1945, as well as in a 1944 paper of Wintner.

The Hardy space of Dirichlet series $\mathscr{H}^2$

Under $\mathscr{S}$, $L^2(0,1)$ is mapped to the following space:

Definition

$\mathscr{H}^2$ consists of all Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

($s = \sigma + it$ a complex variable) with

$$\|f\|_{\mathscr{H}^2}^2 = \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

Since by Cauchy–Schwarz, $|f(s)|^2 \le \|f\|_{\mathscr{H}^2}^2 \sum_{n=1}^{\infty} n^{-2\sigma}$, an f in $\mathscr{H}^2$ is analytic in $\mathbb{C}_{1/2}^+ = \{s = \sigma + it : \ \sigma > 1/2\}$.

Observation on the Riesz basis problem

By the Beurling–Wintner transformation, the following is essentially immediate from the definition of a Riesz basis:

Observation (Hedenmalm–Lindqvist–Seip 97)

The dilated system $\varphi(nx)$ is a Riesz basis for $L^2(0,1)$ if and only if both $\mathscr{S}\varphi$ and $1/\mathscr{S}\varphi$ are multipliers for $\mathscr{H}^2$.

Recall: m a multiplier means m a Dirichlet series such that mf in $\mathscr{H}^2$ whenever f is in $\mathscr{H}^2$. (Then obviously m itself is in $\mathscr{H}^2$ and $f \mapsto mf$ is a bounded operator.)

Reproducing kernel of $\mathscr{H}^2$

Clearly, $\mathscr{H}^2$ is a Hilbert space with reproducing kernel, and the kernel is

$$K_w(s) = \zeta(s + \overline{w}),$$

where ζ is the Riemann zeta-function defined in $\sigma > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}.$$

Thus, for f in $\mathscr{H}^2$ and $\sigma > 1/2$, we have

$$f(s) = \langle f, \zeta(\cdot + \overline{s}) \rangle.$$

Harald Bohr's viewpoint

- Set $z_1 = 2^{-s}$, $z_2 = 3^{-s}$, ..., $z_j = p_j^{-s}$, ..., i.e. think of the "prime powers" p^{-s} as independent variables (by the fundamental theorem of arithmetic).
- In this way, $\mathscr{H}^2$ may be thought of as $H^2(\mathbb{D}^\infty)$.
- A simple estimate using the multiplicative definition of $\zeta(s)$ shows that the series converges in $\mathbb{D}^\infty \cap \ell^2$. ($\mathbb{C}^+_{1/2}$ can be thought of as a subset.)

The theorem on multipliers

We denote the set of multipliers by $\mathscr{M}$. As noted before, every multiplier m defines a bounded operator on $\mathscr{H}^2$; the corresponding operator norm is denoted by $\|m\|_{\mathscr{M}}$.

Theorem (Hedenmalm–Lindqvist–Seip 97)

$$\mathscr{M} = \mathscr{H}^2 \cap H^\infty(\mathbb{C}_0) \text{ and } \|m\|_{\mathscr{M}} = \sup_{s\in\mathbb{C}_0} |m(s)|.$$

- Thus the multipliers live a fully-fledged life in the larger half-plane $\mathbb{C}_0$.
- We write $\mathscr{M} = \mathscr{H}^\infty := \mathscr{H}^2 \cap H^\infty(\mathbb{C}_0)$.
- Our proof uses the Bohr lift and relies on the fact that $\mathscr{B}\mathscr{H}^\infty = H^\infty(\mathbb{T}^\infty)$.
- In hindsight: The multiplier theorem (stated and proved differently) was known to Toeplitz in the 1920s ...

Beurling's completeness problem: Cyclicity in $\mathscr{H}^2$

By the Beurling–Wintner transformation, proving completeness of the system $\varphi(nx)$ in $L^2(0,1)$ amounts to proving cyclicity of $\mathscr{S}\varphi$ in $\mathscr{H}^2$. This is a much more delicate problem, studied by Wintner, Beurling, Kozlov, Helson, Hedenmalm–Lindqvist–S, Nikolski (and possibly others). Only partial results are known. There is (not surprisingly), quantitatively, a huge difference between the two problems (Riesz basis and completeness).

Example

The Davenport series

$$\varphi_\lambda(x) := \sum_{n=1}^{\infty} \frac{\sin(\pi n x)}{n^\lambda}$$

yields a Riesz basis iff $\lambda > 1$ and completeness iff $\lambda > 1/2$.

“In a shadow of RH”

This metaphor is used by Nikolski (2012), in the title of so far the last significant paper on the dilation completeness problem, to illustrate the relation to the Riemann hypothesis (RH).

Indeed, the following statement about dilations of a **nonperiodic** function is equivalent to RH.

The Nyman–Báez-Duarte criterion

Set $\rho(x) := x - [x]$ *and* $\varphi(x) := \rho(1/x)$. *The function* $\chi_{[0,1]}(x)$ *is in the closure in* $L^2(0,\infty)$ *of the system*

$$\varphi(nx), \quad n = 1,2,3,...$$

A number of more direct nontrivial links to the Riemann zeta function and analytic number theory (not just “shadows”) appear in our subject.

H^p spaces of Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$

We follow Bayart (2002) and assume $0 < p < \infty$.

- On the one hand, we define $\mathscr{H}^p$ as the completion of the set of Dirichlet polynomials $P(s) = \sum_{n=1}^{N} a_n n^{-s}$ with respect to the norm (quasi-norm when $0 < p < 1$)

$$\|P\|_{\mathscr{H}^p} = \left(\lim_{T\to\infty} \frac{1}{T}\int_0^T |P(it)|^p dt\right)^{1/p}.$$

- On the other hand, via the Bohr lift $z_j = p_j^{-s}$, $\mathscr{H}^p$ is isometrically isomorphic to $H^p(\mathbb{T}^\infty)$, the closure in $L^p(\mathbb{T}^\infty)$ of holomorphic polynomials in infinitely many variables (z_j).

Explicitly, in multi-index notation we write $n = (p_j)^{\alpha(n)}$ and obtain

$$\mathscr{B}f(z) = \sum_{n=1}^{\infty} a_n z^{\alpha(n)},$$

the Bohr lift of $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Note: $\|f\|_{\mathscr{H}^2}^2 = \sum_n |a_n|^2$.

Overall goals

We want to do the usual things:

- understand the basic function theoretic properties of $\mathscr{H}^p$
- study suitable analogues of classical operators acting on $\mathscr{H}^p$, typically operators defined in terms of symbols and the interplay between properties of the operator and function theoretic properties of the symbol

and in addition to:

- understand how our theory of Hardy spaces $\mathscr{H}^p$ interacts with relevant parts of number theory and possibly other fields.

From one to infinitely many variables

Our theorem on multipliers is an example of a result obtained by looking for analogies with the classical theory of Hardy spaces. It is in line with the perspective of Rudin's classic treatise on complex analysis in polydiscs (from the preface):

"*Briefly, the object is to see how much of our extremely detailed knowledge about holomorphic functions in the unit disc (...) can be carried over to an analogous setting, namely to polydiscs.*"

This is natural and the most obvious approach we may take, and it has led to many interesting results of which only a few will be mentioned in this talk. There are however some less evident points that can be made about the relation to the classical theory of Hardy spaces.

The Riemann zeta function—a reminder

"Additive" definition:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \qquad \mathrm{Re}(s) > 1.$$

"Multiplicative" definition (Euler product)

$$\zeta(s) := \prod_{p} \left(1 - p^{-s}\right)^{-1}, \qquad \mathrm{Re}(s) > 1.$$

The multiplicative representation is what relates $\zeta(s)$ to $\pi(x)$ (the number of primes $p \leq x$), for instance by the formula

$$\log \zeta(s) = s \int_{2}^{\infty} \frac{\pi(x)}{x(x^s - 1)} dx.$$

The motivation for the study of $\zeta(s)$ is our desire to understand the behavior of $\pi(x)$.

Interplay between additive and multiplicative structure

The interplay between the additive and the multiplicative embodied in $\zeta(s)$ appears in our context in the following way:

- Interplay between function theory in half-planes (“additive” model) and function theory on polydiscs (“multiplicative” model)
- Another version of this which is sometimes useful to keep in mind: the dual group (in the Pontryagin sense) of $\mathbb{T}^\infty$ is the multiplicative group $\mathbb{Q}_+$ which has as well an additive structure (the order induced by the numerical size of positive rational numbers).

Although much of the theory developed is inspired by the classical H^p theory, the interplay mentioned above (with no counterpart for classical H^p) plays a central role. Now some examples to illustrate this point and other new features.

Multiplicative and additive Riesz projection

1. $H^2(\mathbb{T}^\infty)$ is a closed subspace of $L^2(\mathbb{T}^\infty)$ The dual group of $\mathbb{T}^\infty$ is the multiplicative group of positive rational numbers $\mathbb{Q}_+$, and $H^2(\mathbb{T}^\infty)$ is the subspace of functions whose Fourier coefficients vanish for non-integers in $\mathbb{Q}_+$. The orthogonal projection $P : L^2(\mathbb{T}^\infty) \to H^2(\mathbb{T}^\infty)$ is the **multiplicative** analogue of one-dimensional Riesz projection. NB! It is unbounded on $L^p(\mathbb{T}^\infty)$ for $p \neq 2$.
2. Since $\mathbb{Q}_+$ is an ordered group (ordered by the numerical size of its members), we may project by restricting the summation of the Fourier series to those q in $\mathbb{Q}_+$ that exceed a fixed value. By a classical theorem of Helson (1957), this **additive** projection enjoys the same L^p boundedness as one-dimensional Riesz projection.

Point evaluations of functions in $\mathscr{H}^p$

- As already noted above, the reproducing kernel K_w of $\mathscr{H}^2$ is $K_w(s) = \zeta(s+\overline{w})$, whence

$$|f(\sigma+it)| \le (\zeta(2\sigma))^{1/2}\|f\|_{\mathscr{H}^2} \le ((\sigma-1/2)^{-1/2}+C)\|f\|_{\mathscr{H}^2}$$

  for every f in $\mathscr{H}^2$ and $\sigma > 1/2$ with C an absolute constant.

- It follows from work of Cole and Gamelin (1986), via the Bohr lift, that this extends to $\mathscr{H}^p$ in the following natural way:

$$|f(\sigma+it)| \le (\zeta(2\sigma))^{1/p}\|f\|_{\mathscr{H}^p} \le ((\sigma-1/2)^{-1/p}+C)\|f\|_{\mathscr{H}^p}.$$

Both of these **multiplicative** estimates are sharp, so functions in $\mathscr{H}^p$ are analytic in the half-plane $\mathbb{C}_{1/2} := \{\sigma+it : \sigma > 1/2\}$ and fails in general to be analytic in any larger domain.

The embedding inequality

By a basic inequality of Montgomery and Vaughan, we have the local embedding inequality

$$\int_{\theta}^{\theta+1} |f(1/2+it)|^2 dt \leq C\|f\|_{\mathscr{H}^2}^2 \qquad (2)$$

for every real number θ, with C an absolute constant independent of θ. What (2) reveals, along with the Bohr lift, is a link between $H^2(\mathbb{T}^\infty)$ and $H^2(\mathbb{C}_{1/2})$.

Embedding problem: Does (2) extend to $\mathscr{H}^p$, i.e., do we have

$$\int_{\theta}^{\theta+1} |f(1/2+it)|^p dt \leq C(p)\|f\|_{\mathscr{H}^p}^p?$$

Partial answer: Holds trivially for $p = 2n$, $n = 1, 2, 3, ...$, and fails for $p < 2$ (breakthrough result of Harper (2017)).

New phenomena 1: The Bohnenblust–Hille inequality

The Bohnenblust–Hille inequality (1931): This is a central result in the function theory of $\mathscr{H}^\infty$. It says that the $\ell^{2m/(m+1)}$ norm of the coefficients of an m-homogeneous polynomial in n complex variables is bounded by a constant $C(m)$ times the supremum of the polynomial in $\mathbb{D}^n$. The remarkable point is that $C(m)$ is independent of the dimension n, and the exponent $2m/(m+1)$ is optimal, i.e., it can not be taken smaller. ($C(m)$ has been subject to much recent work; see Bayart–Pellegrino–Seoane-Sepúlveda (2014) for the latest improvement.)
Thanks to Konyagin–Queffélec (2000), de la Bretèche (2008), Defant–Frerick–Ortega-Cerdá–Ouanies–S (2011):

$$\sup\left\{c:\ \sum_{n=2}^{\infty}|a_n|n^{-1/2}\exp(c\sqrt{\log n\log\log n})<\infty,\ f\in\mathscr{H}^\infty\right\}=1/\sqrt{2}.$$

New phenomena 2: Interaction with probability

A probabilistic model: Think of the variables z_j as independent Steinhaus variables and Haar measure on $\mathbb{T}^\infty$ as a probability measure on the corresponding probability space. This model, first considered in a slightly different guise by Wintner, allows us to understand the interplay between the variables z_j in probabilistic terms, and probabilistic methods become available in our analysis.

Example: The work of Harper (2017) which we will come back to.

Summary so far—and the rest of the talk

The theory of Hardy spaces of Dirichlet series and the associated operator theory is being shaped within this framework of different viewpoints and relations. The rest of this talk will present only the following points:

1. Analogues of some classical inequalities (Hardy and Carleman/Hardy–Littlewood)
2. Multiplicative Hankel forms (following Helson) and the multiplicative Hankel matrix
3. An Ω result for $\zeta(s)$.

Many interesting things will be omitted:

- Zeros sets, interpolating sequences (with quantitative estimates), Fatou theorems, Littlewood–Paley formulas, coefficient estimates; composition operators; Volterra operators; partial sum operators, the multiplicative Riesz projection, GCD sums and Poisson integrals, further relations with systems of dilated functions, ...

Analogues of some classical inequalities

We return to the disc and consider two classical inequalities:

$$\sum_{n=0}^{\infty} \frac{|c_n|}{n+1} \le \pi \|f\|_{H^1(\mathbb{T})} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1} \le \|f\|_{H^1(\mathbb{T})}^2,$$

which are valid for $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in $H^1(\mathbb{T})$. Here the first inequality is known as Hardy's inequality and the second one goes back to Carleman; it is also associated with Hardy–Littlewood. While Hardy's inequality in general is the stronger one, the virtue of the latter is that it is contractive, i.e., the constant on the right-hand side is 1.

Hardy and Carleman for Dirichlet series

The analogues should be, respectively,

$$\sum_{n=2}^{\infty} \frac{|a_n|}{\sqrt{n}\log n} \le C\|f\|_{\mathscr{H}^1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n)} \le \|f\|_{\mathscr{H}^1}^2,$$

where $f(s) = \sum_{n=}^{\infty} a_n n^{-s}$ in $\mathscr{H}^1$ and $d(n)$ is the divisor function. The second inequality is true, thanks to a clever iteration of Carleman's inequality, due to Helson (2006), building on earlier work of Bayart. In other words, it is just an *iteration* of the one variable inequality and thus a **multiplicative** inequality.

The first inequality is *not* known to hold, but I will argue in what follows that *if* it holds, then this **additive** inequality is indeed the analogue of Hardy's inequality.

Multiplicative Hankel forms (following Helson)

In two papers, published in 2006 and (posthumously) in 2010, Henry Helson initiated a study of multiplicative Hankel matrices, which are finite or infinite matrices whose entries $a_{m,n}$ only depend on the product $m \cdot n$.
I will discuss the motivation for these papers, their contents, and a few subsequent developments, in particular:

- Failure of Nehari's theorem
- The analogue of the Hilbert matrix and an interpretation of our conjectured Hardy inequality.

Recent work of Brevig–Perfekt and Perfekt–Pushnitski will not be covered.

Multiplicative Hankel forms

For $\varrho = (\varrho_1, \varrho_2, \varrho_3, \dots)$ in ℓ^2 its corresponding multiplicative Hankel form on $\ell^2 \times \ell^2$ is given by

$$\varrho(a,b) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varrho_{mn} a_m b_n = \sum_{k=1}^{\infty} \varrho_k \sum_{d|k} a_d b_{k/d},$$

which initially is defined at least for finitely supported $a, b \in \ell^2$. If we write $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$, and $\psi(s) = \sum_{n=1}^{\infty} \overline{\varrho}_n n^{-s}$, then we observe that

$$\varrho(a,b) = H_\psi(f,g) := \langle fg, \psi \rangle_{\mathscr{H}^2} = \langle \mathscr{B} f \mathscr{B} g, \mathscr{B} \psi \rangle_{L^2(\mathbb{T}^\infty)}.$$

We say $\psi(s)$ is a symbol of the Hankel form.

Helson's work

The definition of H_ψ on $\mathbb{T}^\infty$ makes sense if $\mathscr{B}\psi$ is replaced by any function in L^∞; we say H_ψ has a bounded symbol if there exists a bounded Φ on $\mathbb{T}^\infty$ such that $H_\psi(f,g) = \langle \mathscr{B}f\mathscr{B}g, \Phi\rangle_{L^2(\mathbb{T}^\infty)}$.

Question (Helson 2010)

Does every bounded multiplicative Hankel form have a bounded symbol ψ on $\mathbb{T}^\infty$?

Equivalently: Do we have $\mathscr{H}^1 = \mathscr{H}^2 \odot \mathscr{H}^2$ (cf. what is known about $H^1(\mathbb{T}^d)$ for $d < \infty$ by results of Ferguson–Lacey (2002) and Lacey–Terwilleger (2009))? In 2006, Helson established the beautiful inequality

$$\left(\sum_{n=1} |a_n|^2 [d(n)]^{-1}\right)^{1/2} \le \|f\|_{\mathscr{H}^1},$$

and used it to show that multiplicative Hankel forms in the Hilbert–Schmidt class S_2 have bounded symbols.

Failure of Nehari's theorem

Based on an idea in Helson's 2010 paper, we found:

Theorem (Ortega-Cerdà–Seip 2013)

There exist bounded multiplicative Hankel forms without bounded symbol.

However, while relying on Helson's basic idea of using the Schur test, we did not use his proposed symbol

$$S_N(s) = \sum_{n=1}^{N} n^{-s}$$

(partial sum of $\zeta(s)$ on the "0-line"). Helson showed that the theorem above would follow if one could prove that

$$\|S_N\|_{\mathscr{H}^1} = o(\sqrt{N}),$$

which became known as Helson's conjecture.

Harper: “Better than square-root cancellation”

Several researchers in analytic number theory believed for some time that Helson’s conjecture should fail.

Surprisingly, however, Helson’s conjecture was recently solved in the most precise way:

Theorem (Adam Harper, arxiv 2017)

We have

$$\|S_N\|_1 \asymp \sqrt{N}/(\log\log N)^{1/4}.$$

This breakthrough is based on an ingenious combination of number theoretic and probabilistic arguments (more precisely: critical approximately Gaussian multiplicative chaos).

Corollary: The embedding inequality fails for $0 < p < 2$.

The classical Hilbert matrix

The prime example of an infinite Hankel matrix is the Hilbert matrix:

$$A := \left(\frac{1}{m+n+1}\right)_{m,n\geq 0}$$

The Hilbert matrix can be viewed as the matrix of the integral operator

$$\mathbf{H}_a f(z) := \int_0^1 f(t)(1-zt)^{-1}dt \qquad (3)$$

with respect to the standard basis $(z^n)_{n\geq 0}$ for the Hardy space $H^2(\mathbb{D})$. Magnus (1950) used this representation to prove that the Hilbert matrix has no eigenvalues and that its continuous spectrum is $[0,\pi]$.

What is the multiplicative Hilbert matrix?

We start from the analogous integral operator:

$$\mathbf{H}f(s) := \int_{1/2}^{+\infty} f(w)(\zeta(w+s)-1)\,dw$$

acting on Dirichlet series $f(s) = \sum_{n=2}^{\infty} a_n n^{-s}$. We say that f is in $\mathscr{H}_0^2$ if f is in $\mathscr{H}^2$ and $a_1 = 0$. The reproducing kernel K_w of $\mathscr{H}_0^2$ is $K_w(s) = \zeta(s+\overline{w})-1$. This implies that

$$\langle \mathbf{H}f, g\rangle_{\mathscr{H}_0^2} = \int_{1/2}^{\infty} f(w)\overline{g(w)}\,dw. \tag{4}$$

This makes sense for all f, g in $\mathscr{H}_0^2$ because arc length measure on $(1/2, +\infty)$ is a Carleson measure for $\mathscr{H}_0^2$. Hence $\mathbf{H}$ is well defined and bounded on $\mathscr{H}_0^2$. Since $\langle f, \mathbf{H}f\rangle_{\mathscr{H}_0^2} = 0$ if and only if $f \equiv 0$. So (4) also implies that $\mathbf{H}$ is strictly positive.

The multiplicative Hilbert matrix

Since

$$\int_{1/2}^{\infty}(nm)^{-w}\,dw = \frac{1}{\sqrt{mn}\log(mn)},$$

the matrix of $\mathbf{H}$ with respect to the basis $(n^{-s})_{n\geq 2}$ is

$$M := \left(\frac{1}{\sqrt{mn}\log(mn)}\right)_{m,n\geq 2}.$$

We call M the **multiplicative Hilbert matrix.**

Theorem (Brevig–Perfekt–Seip–Siskakis–Vukotić 2016)

The operator $\mathbf{H}$ *is a bounded and strictly positive operator on* $\mathscr{H}_0^2$ *with* $\|\mathbf{H}\| = \pi$*. It has no eigenvalues, and the continuous spectrum is* $[0,\pi]$*.*

The symbol of the multiplicative Hilbert matrix

The multiplicative Hankel matrix has analytic symbol

$$\psi(s) := \sum_{n=2}^{\infty} \frac{n^{-s}}{\sqrt{n}\log n}.$$

We observe that $-\psi$ is, up to a linear term, a primitive of the Riemann zeta function. Returning to Nehari, we ask:

Question

Does the multiplicative Hilbert matrix have a bounded symbol?

Equivalently, we may ask whether we have

$$\left| a_1 + \sum_{n=2}^{\infty} \frac{a_n}{\sqrt{n}\log n} \right| \lesssim \|f\|_{\mathscr{H}^1}$$

when $f(s) = \sum a_n n^{-s}$ is in $\mathscr{H}^1$. If we put absolute values on the terms in this sum, this would—if valid—be the right analogue of Hardy's inequality mentioned earlier in the talk.

$\zeta(s)$ on the half-line $\sigma = 1/2$

We have seen "shadows" of $\zeta(s)$ several times (Helson's conjecture, the multiplicative Hilbert matrix). More direct links:

- Computation of pseudomoments (following Conrey and Gamburd 2006), i.e.,

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \left|\sum_{n=1}^{N} n^{-1/2-it}\right|^{2k} dt.$$

  (the right asymptotics still unknown for $0 < k \leq 1/2$)

- Ω theorems, i.e. lower bounds for the growth of $|\zeta(1/2+it)|$ and the argument of $\zeta(1/2+it)$.

I will state one Ω-result. It will take us too far to indicate the proof of this. It builds on several lines of research; the key construction comes from a problem in discrepancy theory related to **systems of dilated functions**, and we used the Bohr lift to solve this problem in an earlier paper.

But first: The Lindelöf hypothesis

The **Lindelöf hypothesis** asserts that $\zeta(1/2+it) = O(|t|^{\varepsilon})$, and, on the Riemann hypothesis, it is known that

$$|\zeta(1/2+it)| \ll \exp\left(c\frac{\log|t|}{\log\log|t|}\right).$$

The best upper bound that has been proved, is

$$|\zeta(1/2+it)| \ll |t|^{13/84+\varepsilon},$$

obtained by Bourgain (2016).

The progress during the last century is a reduction of size $1/84$ in the exponent, from $1/6 = 14/84$ proved by Hardy and Littlewood!

3. But how big do we know that $|\zeta(1/2+it)|$ can be?

Theorem (Bondarenko-S (Duke Math J. 2017); arxiv 2017)

$$\max_{1\le t\le T}\left|\zeta\Big(\frac{1}{2}+it\Big)\right| \ge \exp\left((1+o(1))\sqrt{\frac{\log T\log_3 T}{\log_2 T}}\right), \ T\to\infty.$$

Here the novelty is the triple log:

- Montgomery (assuming RH) and Balasubramanian and Ramachandra (uncondionally) proved in 1977 that there exist arbitrarily large t such that

$$\left|\zeta\Big(\frac{1}{2}+it\Big)\right| \gg \exp\left(c\sqrt{\frac{\log t}{\log\log t}}\right).$$

Farmer–Gonek–Hughes (2007) have conjectured, by use of random matrix theory, that the right bound is

$$\exp((1/\sqrt{2}+o(1))\sqrt{\log t\log\log t}).$$

Recent papers related to this talk

1. A. Bondarenko and K. Seip, *Extreme values of the Riemann zeta function and its argument*, arXiv:1704.06158.
2. A. Bondarenko, O. F. Brevig, E. Saksman, K. Seip, and J. Zhao, *Hardy spaces of Dirichlet series and pseudomoments of the Riemann zeta function*, arXiv:1701.06842.
3. O. F. Brevig, K.-M. Perfekt, and K. Seip, *Volterra operators on Hardy spaces of Dirichlet series*, to appear in J. Reine Angew. Math.
4. A. J. Harper, *Moments of random multiplicative functions, I: Low moments, better than squareroot cancellation, and critical multiplicative chaos*, arXiv:1703.06654.
5. W. Heap, *Upper bounds for L^q norms of Dirichlet polynomials with small q*, arXiv:1703.08842
6. K.-M. Perfekt and A. Pushnitski, *On Helson matrices: moment problems, non-negativity, boundedness, and finite rank*, arXiv:1611.03772.
7. E. Saksman and K. Seip, *Some open questions in analysis for Dirichlet series*, in "Recent progress on operator theory and approximation in spaces of analytic functions", pp. 179–191, Contemp. Math. **679**, Amer. Math. Soc., Providence, RI, 2016.