On the coarse embeddability of Hilbert space

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(jointly with F. Baudier and G. Lancien)

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Theorem (Kasparov and Yu (2006))

If (M, d) is a uniformly locally finite metric space, which coarsely embeds into a super reflexive Banach space, then (M, d) satisfies the coarse geometric Novikov conjecture.

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- $\ell_2 \hookrightarrow X$ coarsely, if X has non trivial co-type, i.e. $\ell_\infty^n \not\subset X$, uniformly, and X has unconditional basis (Ostrovskii),
- every separable metric space bi-Lipschitzly embeds into c_0 (Aharoni).

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- Kalton's Property $\mathcal{Q}$

Theorem (Kalton 2007)

If X has not property $\mathcal{Q}$, then X does not coarsely embed into a reflexive Banach space.

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Hilbert space does not coarsely, embed in T^*, where T^* is the original space constructed by Tsirelson.

We need the following two properties of T^* :

- T^* is reflexive.
- T^* has an unconditional basis (e_i), so that:

  If $n \in \mathbb{N}$ and $(x_j)_{j=1}^n$ is normalized block so that $n \le \min \operatorname{supp}(x_1)$, then $(x_j)_{j=1}^n$ is 2- equivalent to the ℓ_∞^n basis, i.e.

$$\Big\| \sum_{j=1}^n a_j x_j \Big\| \le 2 \max_{j \le n} |a_j| \text{ for all } (a_j)_{j=1}^n \subset \mathbb{R}.$$

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Instead of second condition is would be enough that X is *asymptotically* c_0 in the sense of Maurey, Milman, and Tomczak-Jaegerman.

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A family(X_i, d_i) *equi coarsely* (or *equi uniformly*) embeds into Y, if there exist $\rho, \omega : [0, \infty) \to [0, \infty)$, and $f_i : X_i \to Y$, $i \in I$, so that

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A family(X_i, d_i) *equi coarsely* (or *equi uniformly*) embeds into Y, if there exist $\rho, \omega : [0, \infty) \to [0, \infty)$, and $f_i : X_i \to Y$, $i \in I$, so that
$\rho(t) \leq \rho_{f_i}(t), \omega_{f_i}(t) \leq \omega(t)$
$\rho(t) \to \infty$, $t \to \infty$ and $\omega(t) < \infty$, $t > 0$
(resp. $\rho(t) > 0$, $t > 0$, and $\lim_{t\to 0} \omega(t) = 0$).

Hamming cubes over an infinite alphabet

For $k \in \mathbb{N}$, $[\mathbb{N}]^k = \{\overline{n} \subset \mathbb{N} : \#\overline{n} = k\}$

For $\overline{m} = \{m_1 < m_2 < \cdots < m_k\}$ and $\overline{n} = \{n_1 < n_2 < \cdots < n_k\}$ in $[\mathbb{N}]^k$

$$d_H(\overline{m}, \overline{n}) = \#\{j : m_j \neq n_j\}.$$

$$H_k^\omega = ([\mathbb{N}]^k, d_H)$$

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In order to show our main result, we will show that

- H_k^ω, $k \in \mathbb{N}$, equi coarsely embed into ℓ_2,
- H_k^ω, $k \in \mathbb{N}$, does not equi coarsely embed into T^*.

Proposition

Assume that (e_j) normalized basic sequence in X. Define

$$\Phi(n) := \inf\Big\{\Big\|\sum_{j\in A}\sigma_j e_j\Big\| : A\in[\mathbb{N}]^n, \sigma_j=\pm 1, j\in A\Big\}.$$

[Lower Fundamental Function of (e_j)]

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[Lower Fundamental Function of (e_j)]

Then there is a map $f : [\mathbb{N}]^k \to X$, with

$$\frac{1}{2}\Phi\big(2d_H(\overline{m},\overline{n})\big) \le \big\|f(\overline{m}) - f(\overline{n})\big\| \le d_H(\overline{m},\overline{n}), \quad \overline{m},\overline{n} \in [\mathbb{N}]^k.$$

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Then there is a map $f:[\mathbb{N}]^k\to X$, with

$$\frac{1}{2}\Phi\big(2d_H(\overline{m},\overline{n})\big)\le\big\|f(\overline{m})-f(\overline{n})\big\|\le d_H(\overline{m},\overline{n}),\quad \overline{m},\overline{n}\in[\mathbb{N}]^k.$$

Thus if $\Phi(n)\to\infty$, for $n\to\infty$, then f is coarse embedding.

Remark

It is enough to assume that X has a normalized basic sequence with spreading model not equivalent to the c_0-unit vector basis.

Proof.

Let $b : \{1, 2 \ldots k\} \times \mathbb{N} \to \mathbb{N}$ be a bijection, and define:

$\square$

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$$f(\overline{n}) = \frac{1}{2} \sum_{i=1}^{k} e_{b(i,n_i)}, \text{ for } \overline{n} = \{n_1 < n_2 < \dots < n_k\} \in [\mathbb{N}]^k.$$

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For $\overline{m} = \{m_1 < m_2 < \dots m_k\}$ and $\overline{n} = \{n_1 < n_2 < \dots n_k\}$, we can write

$$\begin{aligned}\|f(\overline{n}) - f(\overline{m})\| &= \Big\|\frac{1}{2}\sum_{i=1, m_i \neq n_i}^{k} (e_{b(i,m_i)} - e_{b(i,n_i)})\Big\| \\ &= \frac{1}{2}\Big\|\sum_{j=1}^{2d_H(\overline{m},\overline{n})} \sigma_j e_{q_j}\Big\| \begin{cases} \leq d_H(\overline{m}, \overline{n}) \\ \geq \frac{1}{2}\Phi(2d_H(\overline{m}, \overline{n})). \end{cases}\end{aligned}$$

with $q_1 < q_2 < \dots q_{2d_H(\overline{m},\overline{n})}$ and $(\sigma_j)_{j=1}^{2d_H(\overline{m},\overline{n})} \subset \{\pm 1\}$.

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with $q_1 < q_2 < \ldots q_{2d_H(\overline{m},\overline{n})}$ and $(\sigma_j)_{j=1}^{2d_H(\overline{m},\overline{n})} \subset \{\pm 1\}$.
Thus f is 1 Lipschitz, and $\|f(\overline{n}) - f(\overline{m})\| \geq \frac{1}{2} \Phi\big(2d(\overline{m}, \overline{n})\big)$. □

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Theorem (Concentration Inequality)

Assume that $f : [\mathbb{N}]^k \to T^*$ *is a map, for which*

$$\mathrm{Lip}(f) = \sup\left\{\left\|f(\overline{m}) - f(\overline{n})\right\| : d_H(\overline{m}, \overline{n}) = 1\right\} = \omega_f(1) < \infty.$$

Then for $\varepsilon > 0$ *there is an infinite* $\mathbb{M} \subset \mathbb{N}$*, so that*

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$$\mathrm{diam}\big(f([\mathbb{M}]^k)\big) \le 4\mathrm{Lip}(f) + \varepsilon.$$

But note that d_H-diam$([\mathbb{M}]^k) = k \nearrow \infty$.

Thus, the H_k^∞, $k \in \mathbb{N}$, are not equi- coarsely embeddable into T^*.

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Using reflexivity choose for $n_1 < n_2 < \dots n_{k-1}$ in $\mathbb{N}$, infinite $N = N(n_1, n_2, \dots, n_{k-1}) \subset \mathbb{N}$ so that

$$f(n_1, n_2, \dots n_{k-1}) := w - \lim_{n \in N} f(n_1, n_2, \dots, n_{k-1}, n)$$

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exists. Weak lower semi continuity of $\|\cdot\|$:

$$\|f(n_1, n_2, \ldots, n_{k-1}, n) - f(n_1, n_2, \ldots n_{k-1})\| \leq \mathrm{Lip}(f), \quad n \in N.$$

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Iterating and diagonalization: $\exists N \subset \mathbb{N}\, \forall n_1 < n_2 < \dots < n_{k-1}$ in N

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Weak lower semi continuity of $\|\cdot\|$: $\mathrm{Lip}(f|_{[N]^{k-1}}) \le \mathrm{Lip}(f|_{[\mathbb{N}]^k})$.

Iterating previous argument:$\exists\ M \subset \mathbb{N}$ infinite and a family

$$\big(f(m_1, m_2, \dots, m_l) : 0 \le l \le k, m_1 < m_2 < \dots < m_l \text{ in } M\big),$$

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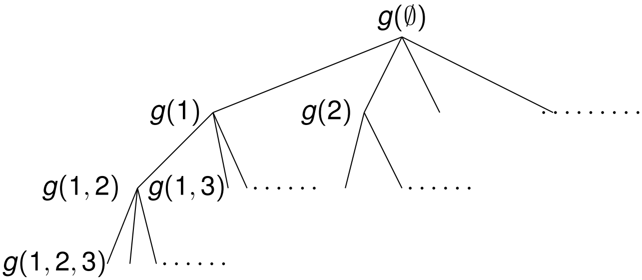
Write $M = \{m_1 < m_2 < m_3 \ldots\}$, and relable

$$g(n_1, n_2, \ldots, n_l) = f(m_{n_1}, m_{n_2}, \ldots, m_{n_l})$$

for $0 \leq l \leq k$ and $n_1 < n_2 < \ldots < n_l$.

Weakly converging tree



Write

$$g(m_1, m_2, \ldots m_l) = g(\emptyset) + \sum_{j=1}^{l} \underbrace{g(m_1, m_2, \ldots m_j) - g(m_1, m_2, \ldots m_{j-1})}_{z(m_1, m_2, \ldots m_j) \in \mathrm{Lip(f)} B_{T*}}.$$

and $z(\emptyset) = g(\emptyset)$.

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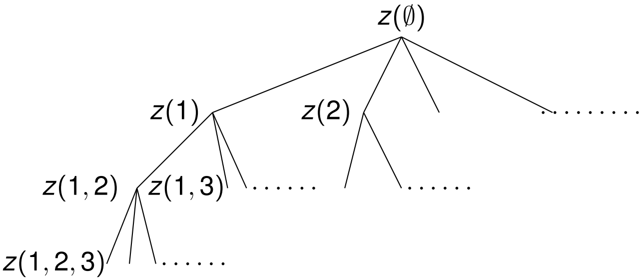
and $z(\emptyset) = g(\emptyset)$.

Thus the family $(z(m_1, m_2, \dots, m_l) : 0 \le l \le k, m_1 < m_2 < \dots m_l)$ is a *weakly null tree Lip*$(f)B_{T*}$, meaning that

$$w - \lim_{m \to \infty} z(m_1, m_2, \dots, m_l) = 0, \text{ for all } l < k, \text{ and } m_1 < m_2 < \dots < m_l.$$

Weakly null tree



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Every weak null sequence (u_n) in T^* has a subsequence which is an arbitrary perturbation of a block sequence. Using this fact we find an infinite $M \subset \mathbb{N}$ and a block tree (all nodes and all branches are block sequences)

$$\big(w(n_1, n_2, \dots n_l) : 0 \leq l \leq k, n_1 < n_2 < \dots < n_l \text{ in } M)\big)$$

with

$$\|z(n_1, n_2, \dots n_l) - w(n_1, n_2, \dots n_l)\| \leq \varepsilon/2k.$$

if $l \leq k$ and $n_1 < n_2 < \dots n_l$ are in M.

Every weak null sequence (u_n) in T^* has a subsequence which is an arbitrary perturbation of a block sequence. Using this fact we find an infinite $M \subset \mathbb{N}$ and a block tree (all nodes and all branches are block sequences)

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Thus by second property of T^*, it follows for $\overline{m}, \overline{n} \in [M]^k$

$$\|g(\overline{m}) - g(\overline{n})\| = \Big\| \sum_{l=1}^{k} z(m_1, m_2, \dots m_l) - z(n_1, n_2, \dots, n_l) \Big\|$$

$$\le \varepsilon + \Big\| \sum_{l=1}^{k} w(m_1, m_2, \dots m_l) \Big\| + \Big\| \sum_{l=1}^{k} w(n_1, n_2, \dots n_l) \Big\| \le 4\mathrm{Lip}(f) + \varepsilon.$$

Uniform Embeddings

With similar proof:

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Theorem (BLS)

H_k^ω, $k \in \mathbb{N}$, equi uniformly embed into B_{ℓ_2}, but not in B_{T^*}.
Thus B_{ℓ_2} does not uniformly embed into B_{T^*}.

Corollary

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Proof.

Let X be non reflexive.

Assume that $F : X \to T^*$ is a coarse embedding.

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By James '72, $\exists (x_k) \subset B_X$:

$$\forall k \in \mathbb{N}\, \forall n_1 < n_2 < \ldots < n_{2k} \quad \Big\| \sum_{j=1}^{k} x_{n_j} - \sum_{j=k+1}^{2k} x_{n_j} \Big\| \geq \frac{k}{2}.$$

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Define $\phi_k : [\mathbb{N}]^k \ni \{n_1 < n_2 < \ldots n_k\} \mapsto \sum_{j=1}^{k} x_{n_j} \in X$, $\mathrm{Lip}(\phi_k) \leq 2$ and $\mathrm{Lip}(F \circ \phi_k) \leq w_F(2)$ and thus

$\exists \mathbb{M} \subset \mathbb{N}$ inf., so that $\mathrm{diam}\big(F \circ \phi_k([\mathbb{M}]^k)\big) < 5 w_F(2)$.

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If $\overline{m} = \{m_1 < m_2 < \ldots m_k\}$ and $\overline{n} = \{n_1 < \ldots n_k\}$ in $\mathbb{M}$, with $m_k < n_1$,

$\|F \circ \phi_k(\overline{m}) - F \circ \phi_k(\overline{n})\| \leq \rho_F\Big(\Big\| \sum_{j=1}^{k} x_{m_j} - \sum_{j=1}^{k} x_{n_j} \Big\|\Big) \leq \rho_F(\frac{k}{2}) \nearrow \infty$, if $k \nearrow \infty$. (which is a contradiction). □

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Assume that Y is coarsely minimal Banach space, and thus coarsely embeds into ℓ_2, which implies by Mendel, Naor that Y must have co-type 2.

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Assume that Y is coarsely minimal Banach space, and thus coarsely embeds into ℓ_2, which implies by Mendel, Naor that Y must have co-type 2.

Since Y also coarsely embeds into T^* it must also be reflexive by previous Corollary.

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Since Y also coarsely embeds into T^* it must also be reflexive by previous Corollary. Secondly, H_k^∞, do not equi coarsely embed into Y, and therefore the spreading model generated by any basic sequence in Y must be isomorphic to c_0. In particular the ℓ_∞^n's are finitely represented, which contradicts that the cotype of Y must be 2. □

Open Problems

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