Lifting of nest approximation properties and related principles of local reflexivity

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"Life in Banach spaces with certain approximation properties is much easier." (Pietsch, History ..., 2007, p. 287)

Recall: A Banach space X has AP: $\exists (S_\nu) \subset \mathcal{F}(X, X)$ such that $S_\nu \to I_X$ uniformly on compact subsets of X.
X has λ-BAP: $\exists (S_\nu)$ as above, and $\|S_\nu\| \leq \lambda \quad \forall \nu$
BAP: λ-BAP $(\exists \lambda)$; MAP= 1-BAP

$\exists$ basis $\Rightarrow$ BAP; $\exists$ monotone basis $\Rightarrow$ MAP

MAP $\Rightarrow$ BAP $\Rightarrow$ AP in X

Converses do not hold even in $X \subset c_0$:
Johnson, Schechtman 1996, Godefroy, Saphar 1988;
Figiel, Johnson, Pełczyński 2011

AP $\not\Rightarrow$ BAP $\not\Rightarrow$ MAP in X

Problem 1 (goes back to Grothendieck's Memoir 1955) :
Does AP $\Rightarrow$ MAP in X^*? "AP $\Rightarrow$ MAP problem"

Grothendieck: "Yes" for X reflexive or X^* separable

This mysterious X^* ...

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Some recent variants of (B)AP:

- weaker forms, e.g. weak MAP [Lima, Oja; Math. Ann. 2005]
  for this direction, see [Lassalle, Oja, Turco; J. Appr. Th. 2016]

- •• stronger forms = refinements of the classical APs:
  APs of pairs [Figiel, Johnson, Pełczyński; Israel J. Math. 2011]
  nest APs [Figiel, Johnson; J. Funct. Anal. 2016]

Our topic: lifting of nest APs between X and X^*

Classics (Grothendieck, Johnson; Enflo, James, Lindenstrauss):

X^*– (λ-B)AP $\Rightarrow$ X – (λ-B)AP $\not\Rightarrow$ X^* – AP

X^* – MAP $\Leftarrow$ X – MAP in all equivalent norms

Problem 2: ? $\Rightarrow$? What about $X = \ell_1$?

- Pr. 1 "Yes" $\Rightarrow$ Pr. 2 "Yes"

[LO2005]: X^*– AP $\Leftrightarrow$ X – weak MAP in all equivalent norms

[Oja2006]: $\Updownarrow$ if X or X^* Asplund

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Classics ((Grothendieck), Reinov; many different proofs):
If X or X^* Asplund, then X^* – AP $\Rightarrow$ X^* – MAP

Problem 3 [LO2005]: Does weak MAP $\Rightarrow$ MAP?

- Pr. 3 "Yes" $\Rightarrow$ Pr. 1 "Yes"

Nests

X – Banach space, $\mathcal{N} \subset \{U \subset X : U - \text{closed subspace}\}$

Recall: $\mathcal{N}$ is a nest in X: $\mathcal{N}$ is linearly ordered by "$\subset$".
(For $U, W \in \mathcal{N}$, one has $U \subset W$ or $W \subset U$.)

Examples
• $\mathcal{N} = \{U\}$
• $\mathcal{N} = \{U_1, U_2, \ldots, U_n\}$, *e.g.*, $U_1 \subset U_2 \subset \ldots \subset U_n$
• U with a basis $e_1, e_2, \ldots$, $U_n := \text{span}\{e_1, e_2, \ldots, e_n\}$
$\mathcal{N} = \{\{0\}, U, X, U_n : n \in \mathbb{N}\}$ or $\mathcal{N} = \{U_n : n \in \mathbb{N}\}$
• Volterra nest $\mathcal{V}$ in $L_p[0,1]$, $1 \leq p < \infty$, $\mathcal{V} = \{L_p[0,t] : 0 \leq t \leq 1\}$

$\mathcal{N}$ is complete: $\{0\}, X \in \mathcal{N}$, and $\mathcal{N}$ is closed under intersections and closures of unions.

Nest approximation properties

Recall: X has (λ-B)AP: $\exists (S_\nu) \subset \mathcal{F}(X, X)$ such that $S_\nu \to I_X$ uniformly on compact subsets of X (and $\|S_\nu\| \leq \lambda$).

$U \subset X$, U – closed subspace
The pair (X, U) has (λ-B)AP: $S_\nu(U) \subset U$ for all ν.
E.g.: X has AP $\Leftrightarrow$ (X, X) has AP $\Leftrightarrow$ $(X, \{0\})$ has AP

$\mathcal{N}$ – nest in X
The pair $(X, \mathcal{N})$ has (λ-B)AP: $S_\nu(U) \subset U$ for all ν and $U \in \mathcal{N}$.
E.g.: (X, U) has AP $\Leftrightarrow$ $(X, \mathcal{N})$ has AP, where $\mathcal{N} = \{U\}$

(X, U) has AP $\Rightarrow$ $X, U, X/U$ all have AP.
Problem 4: ? $\Leftarrow$?

[FJP2011]: X^* – λ-BAP $\Leftrightarrow$ (X, U) – λ-BAP for all U, codim $U < \infty$.

[J. A. Erdos; J. London Math. Soc. 1968]:
$(H, \mathcal{N})$ has AP for all nests $\mathcal{N}$ in a Hilbert space H.

[FJ2016]: For X over $\mathbb{C}$,
$(X, \mathcal{N})$ has AP for all $\mathcal{N} \Leftrightarrow X$ has the Lidskii trace property:
If a nuclear op. T on X has absolutely summable eigenvalues λ_n, then trace $T = \sum_n \lambda_n$.

(X, U) has AP for all $U \Rightarrow X$ has the hereditary AP (HAP)

Problem 5 (Johnson, Szankowski; Ann. Math. 2012, Isr. J. Math. 2014): Does HAP $\Rightarrow$ Lidskii trace property?

Problem 6: Does (X, U) has AP for all $U \Rightarrow (X, \mathcal{N})$ has AP for all $\mathcal{N}$?

- Pr. 5 "Yes" $\Rightarrow$ Pr. 6 "Yes"

Lifting “down” classics: $X^* - (\lambda\text{-B})\text{AP} \Rightarrow X - (\lambda\text{-B})\text{AP}$

General reason: (1) PLR + (2) convex combination argument
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(2) Convex combinations of S_ν give $(\lambda$-B)AP of X. This works for convex APs.

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$\mathcal{A} \subset \mathcal{L}(X, X)$, $\mathcal{A}$ – convex.
• X has $\mathcal{A}$-AP: $\exists (S_\nu) \subset \mathcal{A}$ such that $S_\nu \to I_X$ uniformly on compact subsets of X, i.e., in τ_c of X.
• X^* has $\mathcal{A}$-AP with conjugate operators: $\exists (S_\nu) \subset \mathcal{A}$ such that $S_\nu^* \to I_{X^*}$ in τ_c of X^*.

Example: (λ-B)AP of $(X, \mathcal{N}) = \mathcal{A}$-AP with
$\mathcal{A} = \{S \in \mathcal{F}(X, X) : S(U) \subset U \ \forall U \in \mathcal{N} \quad (\|S\| \leq \lambda)\}$

(2) X^* – $\mathcal{A}$-AP with conjugate op-rs $\Rightarrow$ X – $\mathcal{A}$-AP [Lissitsin, Oja; JMAA 2011]

(2) X^* – $\mathcal{A}$-AP with conjugate operators $\Rightarrow$ X – $\mathcal{A}$-AP

$\mathcal{N}$ – nest in $X \Rightarrow \mathcal{N}^\perp := \{U^\perp : U \in \mathcal{N}\}$ – nest in X^*
$S(U) \subset U \Leftrightarrow S^*(U^\perp) \subset U^\perp$. Hence, in particular,

(2′) $(X^*, \mathcal{N}^\perp)$ – (λ-B)AP with conjugate op-rs $\Rightarrow$ $(X, \mathcal{N})$ – (λ-B)AP

All we need: (λ-B)AP of $(X^*, \mathcal{N}^\perp)$ to be given with conjugate op-rs!
In particular, (λ-B)AP of $(X^*, U^\perp)$ to be given with conjugate op-rs!
Recall (1): PLR works for $(X^*, \{0\}^\perp) = (X^*, X^*)$ = (λ-B)AP of X^*.
Find a “working” PLR (respecting nests) for (λ-B)AP of $(X^*, \mathcal{N}^\perp)$!

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Find a “working” PLR (respecting nests) for (λ-B)AP of $(X^*, \mathcal{N}^\perp)$!

[Oja; Adv. Math. 2014]: PLR respecting subspaces $\Rightarrow$
$\Rightarrow$ (λ-B)AP (with projections) of $(X^*, U^\perp)$ is always given
with conjugate operators (projections).

- $(X^*, U^\perp)$ – (λ-B)AP $\Rightarrow$ (X, U) – (λ-B)AP (by (2′))

PLR respecting subspaces [O2014]: X, Y – Banach spaces; $U \subset X, V \subset Y$ – closed subspaces. Let $S \in \mathcal{F}(Y^*, X^*)$ satisfy $S(V^\perp) \subset U^\perp$. If $F \subset Y^*$, $\dim F < \infty$, and $\varepsilon > 0$, then $\exists T \in \mathcal{F}(X, Y)$ satisfying $T(U) \subset V$ such that

- 1° $\big|\|T\| - \|S\|\big| < \varepsilon$,
- 2° $T^* y^* = S y^*$, $y^* \in F$,
- $2^{\circ\circ}$ $\operatorname{ran} T^* = \operatorname{ran} S$,
- 3° $T^{**} x^{**} = S^* x^{**}$ whenever $S^* x^{**} \in Y$.

When $Y = X$ and S is a projection, also T is a projection.

Applying PLR resp. subsp. twice:

• Finite-rank operators/projections between bi-duals, who respect bi-annihilators, are “locally” bi-conjugate to operators/projections, who respect subspaces.

PLR respecting subspaces $\Rightarrow$ (λ-B)AP (with projections) of $(X^*, U^\perp)$ is always given with conjugate operators (projections).

- $(X^*, U^\perp)$ – (λ-B)AP $\Rightarrow$ (X, U) – (λ-B)AP (by (2′))

Recall: (2′) $(X^*, \mathcal{N}^\perp)$ – (λ-B)AP with conj. op-rs $\Rightarrow$ $(X, \mathcal{N})$ – (λ-B)AP

- $(X^*, U^\perp)$ – λ-BAP with proj. $\Rightarrow$ (X, U) – $(\lambda^2 + 2\lambda)$-BAP with proj.
- $(X^{**}, U^{\perp\perp})$ – λ-BAP with proj. $\Rightarrow$ (X, U) – λ-BAP with proj.

PLR respecting subspaces $\Rightarrow$ (λ-B)AP (with projections) of $(X^*, U^\perp)$ is always given with conjugate operators (projections).

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Recall: (2′) $(X^*, \mathcal{N}^\perp)$ – (λ-B)AP with conj. op-rs $\Rightarrow$ $(X, \mathcal{N})$ – (λ-B)AP

• $(X^*, U^\perp)$ – λ-BAP with proj. $\Rightarrow$ (X, U) – $(\lambda^2 + 2\lambda)$-BAP with proj.
• $(X^{**}, U^{\perp\perp})$ – λ-BAP with proj. $\Rightarrow$ (X, U) – λ-BAP with proj.

NoLiFA: [Chávez-Domínguez; arXiv 2017] uses PLR resp. subsp.:
• If X is separable and if (X, U) has the Lipschitz lifting property (i.e. $\exists R \in \mathcal{L}(X, F(X)), \beta R = I_X$ (β – can. quotient map), $R(U) \subset F(U)$), then: (X, U) – Lipschitz BAP $\Rightarrow$ (X, U) – BAP

[Godefroy, Kalton; Studia Math. 2003]: For arbitrary X:
X – Lipschitz λ-BAP $\Rightarrow$ X – λ-BAP

X, Y – Banach spaces; $\mathcal{U}$ – nest in X, $\mathcal{N}_{\mathcal{U}} = \{V_U : U \in \mathcal{U}\}$ – nest in Y. $\mathcal{N}_{\mathcal{U}}$ is increasing on $\mathcal{U}$: if $U \subset W$ and $U \neq W$ in $\mathcal{U}$, then $V_U \subset V_W$ and $V_U \neq V_W$.

PLR respecting nests [Oja, Veidenberg; J. Funct. Anal. 2017]:
Assume $\{0\} \in \mathcal{U}, \mathcal{N}_{\mathcal{U}}^{\perp}$ is complete. Let $S \in \mathcal{F}(Y^*, X^*)$ satisfy $S(V_U^{\perp}) \subset U^{\perp}$ for all $U \in \mathcal{U}$. If $K \subset X^{**}, L \subset Y^*$ are compact sets, and $\varepsilon > 0$, then $\exists T \in \mathcal{F}(X, Y)$ satisfying $T(U) \subset V_U$ for all $U \in \mathcal{U}$ such that

1° $\big|\|T\| - \|S\|\big| < \varepsilon$,
2° $\|T^* y^* - S y^*\| < \varepsilon$ for all $y^* \in L$,
3° $\|T^{**} x^{**} - S^* x^{**}\| < \varepsilon$ whenever $x^{**} \in K$ and $S^* x^{**} \in Y$.

Applying PLR resp. nests twice (assuming $\mathcal{U}^{\perp\perp}$ is complete):
• Finite-rank operators between bi-duals, who respect bi-annihilators, are “locally” bi-conjugate to operators, who respect subspaces.

PLR respecting nests easily follows from:

Lemma: $\mathcal{G}$ – nest in $X^*, \mathcal{N}_{\mathcal{G}} = \{V_G : G \in \mathcal{G}\}$ – nest in Y, $\mathcal{N}_{\mathcal{G}}$ increasing on $\mathcal{G}$. Assume $\{0\} \in \mathcal{G}$, $\mathcal{N}_{\mathcal{G}}{}^{\perp}$ is complete. Let $T \in X \otimes Y^{**}$ satisfy $T(G) \subset V_G^{\perp\perp}$ for all $G \in \mathcal{G}$. Then $\exists (T_\alpha) \subset X \otimes Y$ satisfying $T_\alpha(G) \subset V_G$ for all α and $G \in \mathcal{G}$ such that

1° $\|T_\alpha\| \to \|T\|$,

2° $T_\alpha^* y^* \to T^* y^*$ for all $y^* \in Y^*$,

3° $T_\alpha x^* \to T x^*$ for those $x^* \in X^*$ for which $T x^* \in Y$.

Sketch of proof:

$$\mathcal{R} := \{R \in X \otimes Y : R(G) \subset V_G \quad \forall G \in \mathcal{G}\}$$
$$\mathcal{S} := \{S \in X \otimes Y^{**} : S(G) \subset V_G^{\perp\perp} \quad \forall G \in \mathcal{G}\}$$
$$\mathcal{R}^{\perp} \subset (X \otimes Y)^* = \mathcal{I}(X, Y^*) \xrightarrow{J} \mathcal{I}(X, Y^{***}) = (X \otimes Y^{**})^* \supset \mathcal{S}^{\perp},$$

where $J(A) = j_{Y^*} A$. Using extension of Ringrose th., $J(\mathcal{R}^{\perp}) \subset \mathcal{S}^{\perp}$.

Define $\Phi(A + \mathcal{R}^{\perp}) = J(A) + \mathcal{S}^{\perp},\ A \in \mathcal{I}(X, Y^*)$,

$$\mathcal{R}^* = \mathcal{I}(X, Y^*)/\mathcal{R}^{\perp} \xrightarrow{\Phi} \mathcal{I}(X, Y^{***})/\mathcal{S}^{\perp} = \mathcal{S}^*.$$

$T \in \mathcal{S}$ and $\Phi^*(T) \in \|T\| B_{\mathcal{R}^{**}} \Rightarrow \exists (T_\alpha) \subset \mathcal{R}$, also 1°, 2°, 3° hold.

Extension of the Ringrose theorem [OV17]:

X, Y – Banach spaces; $\mathcal{G}$ – nest in X^*,
$\mathcal{N}_{\mathcal{G}} = \{V_G : G \in \mathcal{G}\}$ – nest in Y, $\mathcal{N}_{\mathcal{G}}$ increasing on $\mathcal{G}$.
Assume $\{0\} \in \mathcal{G}$, $Y \in \mathcal{N}_{\mathcal{G}}$, $\mathcal{N}_{\mathcal{G}}$ is closed under intersections.
$\mathcal{R} := \{R \in X \otimes Y : R(G) \subset V_G \quad \forall G \in \mathcal{G}\}$. Then :

- $R = x \otimes y \in \mathcal{R} \Leftrightarrow \exists G \in \mathcal{G}$ such that $x \in (G\_)_{\perp}$ and $y \in V_G$.
- Let $R \in X \otimes Y$ have rank n. $R \in \mathcal{R} \Rightarrow R = \sum_{k=1}^{n} x_k \otimes y_k$, $x_k \otimes y_k \in \mathcal{R}$.

$(G\_ := \bigcup\{H \in \mathcal{G} : H \subset G, H \neq G\}$ if $G \neq \{0\}; \quad \{0\}\_ := \{0\}.)$

Ringrose ([Proc. London Math. Soc. 1965] and [Erdos 1968]):
$X^* = Y = H$ – Hilbert space, $\quad \mathcal{N}_{\mathcal{G}} = \mathcal{G}$ – complete nest.

$\mathcal{N}$ – nest in X, assume $\mathcal{N}^{\perp}$ is complete. PLR resp. nests $\Rightarrow$
$\Rightarrow$ (λ-B)AP of $(X^*, \mathcal{N}^{\perp})$ is given with conjugate operators.

• $(X^*, \mathcal{N}^{\perp})$ – (λ-B)AP $\Rightarrow$ $(X, \mathcal{N})$ – (λ-B)AP (by (2′))

Summary: Nest APs nicely "go down".

$\mathcal{N}$ – nest in *X*, assume $\mathcal{N}^\perp$ is complete. PLR resp. nests $\Rightarrow$
$\Rightarrow$ (λ-B)AP of ($X^*, \mathcal{N}^\perp$) is given with conjugate operators.
• ($X^*, \mathcal{N}^\perp$) – (λ-B)AP $\Rightarrow$ ($X, \mathcal{N}$) – (λ-B)AP (by (2′))

Summary: Nest APs nicely "go down".

Lifting "up" from *X* to X^* is known in special cases:

• *X* has the unique extension property; then
• *X* – MAP $\Rightarrow$ X^* – MAP [Godefroy, Saphar; Illin. J. Math. 1988]

• *X* is extendably LR (ELR) [Johnson, Oikhberg; Illin. J. Math. 2001]:
• *X* – λ-ELR, *X* – μ-BAP $\Rightarrow$ X^* – $\lambda\mu$-BAP

(E, F, ε): $E \subset X^{**}, \dim E < \infty; \quad F \subset X^{*}, \dim F < \infty; \quad \varepsilon > 0$

Recall: X is λ-extendably LR: $\forall (E, F, \varepsilon)\, \exists T \in \mathcal{L}(X^{**})$,
$T(E) \subset X, \|T\| \leq \lambda + \varepsilon$,
$x^{*}(Tx^{**}) = x^{**}(x^{*})$ whenever $x^{**} \in E, x^{*} \in F$.

[OV2017]: $(X, \mathcal{N})$ is λ-ELR: $\forall (E, F, \varepsilon)\, \exists T \in \mathcal{L}(X^{**})$,
$T(E) \subset X, \|T\| \leq \lambda + \varepsilon$, $T(U^{\perp\perp}) \subset U^{\perp\perp}$ for all $U \in \mathcal{N}$,
$|x^{*}(Tx^{**}) - x^{**}(x^{*})| \leq \varepsilon$ whenever $x^{**} \in S_E, x^{*} \in S_F$.

Using PLR resp. nests and [Oja, Veidenberg; JMAA 2016] gives
[OV2017]: Let $\mathcal{N}^{\perp\perp}$ be complete. Then
- $(X, \mathcal{N})$ – λ-ELR, $(X, \mathcal{N})$ – μ-BAP $\Rightarrow$ $(X^{*}, \mathcal{N}^{\perp})$ – $\lambda\mu$-BAP.

Rosenthal (see [JO2001]): X^{*} – λ-BAP $\Rightarrow$ X – λ-ELR.
Using PLR resp. nests: [OV2017]: Let $\mathcal{N}^{\perp}$ be complete. Then
- $(X^{*}, \mathcal{N}^{\perp})$ – λ-BAP $\Rightarrow$ $(X, \mathcal{N})$ – λ-ELR.

• Below, let X have the unique extension property:
the only $T \in \mathcal{L}(X^{**})$ such that $\|T\| \leq 1$ and $T|_X = I_X$ is $T = I_{X^{**}}$.

[GS1988]: X – MAP $\Rightarrow$ X^* – MAP. (The same for compact MAP.)

[Oja; JMAA 2006] $\Rightarrow$ $(X, \mathcal{N})$ – weakly compact MAP $\Rightarrow$
$\Rightarrow$ $(X^*, \mathcal{N}^\perp)$ – weakly compact MAP with conj. op-rs.

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[Oja; JMAA 2006] $\Rightarrow$ $(X, \mathcal{N})$ – weakly compact MAP $\Rightarrow$
$\Rightarrow$ $(X^*, \mathcal{N}^\perp)$ – weakly compact MAP with conj. op-rs.

• Lifting "up" of bounded nest APs from X to X^* is possible whenever X already enjoys a weaker *metric* nest AP:

[OV2017]: Assume $(X, \mathcal{N})$ has weakly compact MAP. Then
• $(X, \mathcal{N})$ – λ-BAP $\Rightarrow$ $(X^*, \mathcal{N}^\perp)$ – λ-BAP.

Proof. $(X^*, \mathcal{N}^\perp)$ has weakly compact MAP with conj. op-rs. By [OV2017], $(X, \mathcal{N})$ is 1-ELR of stronger type (with ELR operator $T = S^{**}$, $S \in \mathcal{W}(X, X)$). Since X has also λ-BAP, $(X^*, \mathcal{N}^\perp)$ has $(1 \cdot \lambda)$-BAP by an extension of the Johnson–Oikhberg theorem.