Distortion of Lipschitz Functions on $c_0(\Gamma)$

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Introduction

Definition

Let $X$ be a Banach space and $f : S_X \to \mathbb{R}$. We say $f$ is oscillation stable if for every infinite dimensional subspace $Z \subset X$ and every $\varepsilon > 0$ there exists an infinite dimensional subspace $Y \subset Z$ such that $|f(x) - f(y)| \leq \varepsilon$ for every $x, y \in S_Y$. 
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Definition

A function $f : S_X \to \mathbb{R}$ is said to be **distorted** if there exists an $\varepsilon > 0$ such that for every infinite dimensional subspace $Y$ of $X$ there exist $x, y \in S_Y$ such that $|f(x) - f(y)| > \varepsilon$. 
Theorem (Gowers)
Every Lipschitz function $f: c_0 \rightarrow \mathbb{R}$ is oscillation stable.

Theorem (Odell, Schlumprecht)
There is a distorted Lipschitz function on $\ell_1$. For every $1 < p < \infty$, there is a distorted equivalent norm on $\ell_p$. 
**Theorem (Gowers)**

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Nonseparable case

Definition

Let \((X, \|\cdot\|)\) be a Banach space with a symmetric (possibly uncountable) Schauder basis \(\{e_\gamma\}_{\gamma \in \Gamma}\), where \(\Gamma\) is any nonempty set. We say that a function \(f: X \to \mathbb{R}\) is symmetric if the value \(f(x)\) is preserved under any permutation of the coordinates of \(x\).

Theorem (Hájek, N.)

There is a \(1\)-Lipschitz symmetric function \(F: S_{c_0}(\Gamma) \to \mathbb{R}\), taking values in \([0, 1]\), such that for every nonseparable subspace \(Y \subseteq c_0(\Gamma)\) there are points \(x, y \in S_Y\) such that \(|F(x) - F(y)| > \frac{1}{4}\).
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On $c_{00}(\omega_1)$ define equivalence $x \sim y$ whenever $|\text{supp } x| = |\text{supp } y|$ and there exists a bijection $f : \text{supp } x \rightarrow \text{supp } y$ such that $x(\gamma) = y(f(\gamma))$. We call every equivalence class $[x] \in X := c_{00}(\omega_1)/\sim$ a shape.
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Notation

Let us denote by $L = \{S_i\}_{i=1}^{\infty}$ the sequence of all shapes of norm one with finite support and rational coordinates.
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Lemma (Modified extension formula)

Suppose \((M,d)\) is a metric space and \( g : S \to \mathbb{R} \) a \( K \)-Lipschitz function on some \( S \subseteq M \), taking values only in the interval \([0,1]\). Then the following formula defines a \( K \)-Lipschitz function \( \overline{g} : M \to \mathbb{R} \), taking values only in \([0,1]\) such that \( \overline{g}|_S = g \).

\[
\overline{g}(x) = \min \left\{ \inf_{y \in S} \{g(y) + Kd(x,y)\} , 1 \right\} . \tag{1}
\]
Thank you for your attention.