A project in Complex and non - linear Functional Analysis

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Let X be a space of holomorphic functions on a domain Ω in one or several complex variables. For most of the spaces X the following holds. If for every bump V of Ω, there is a function f_V in X, which is not holomorphically extendable to V, then there is a function f in X non - extendable to any V. That is, Ω is an X - domain of holomorphy ([1], [2], [3]). Further, the set S of such functions f is dense and G_δ in X ([2], [3]). A similar phenomenon holds for functions in X which are totally unbounded on $\partial\Omega$ ([4], [1]).

Let me add some details. If Ω is a domain in $\mathbb{C}$ or $\mathbb{C}^k$ or more generally in a complex Banach space, a bump of Ω is a pair $V = (b_1, b_2)$ where b_1 and b_2 are open balls such that $b_1 \subseteq \overline{b_1} \subseteq \Omega \cap b_2$ and $b_2 \cap \Omega^c \neq \emptyset$. A holomorphic function f on Ω is called extendable if there exists a bump $V = (b_1, b_2)$ of Ω and a bounded holomorphic function $F : b_2 \to \mathbb{C}$ such that $f_{|b_1} = F_{|b_1}$. Otherwise, the function f is called non - extendable.

Let X be a family of holomorphic functions in Ω. Then Ω is called an X - weak domain of holomorphy if for every bump $V = (b_1, b_2)$ of Ω there exists a function $f_V \in X$, such that, $f_{V|b_1}$ does not admit any bounded holomorphic extension on b_2.

The set Ω is called an X - domain of holomorphy if there exists a non - extendable function $f \in X$. Obviously, if Ω is an X - domain of holomorphy, then it is also an X - weak domain of holomorphy and in this case the function f_V may be chosen to be the same for every bump V. Under the following assumptions on X, the converse also holds.

Assumptions on X: Let $X \subseteq H(\Omega)$. We assume that X endowed with the usual operations $+, \cdot$ is a separable topological (complex) vector space whose topology is endowed by a complete metric. Furthermore, if $f \in X$ and $\{f_n\}_{n\geq 1} \subseteq X$ are such that $f_n \to f$ in the topology of X it follows the pointwise convergence $f_n(z) \to f(z)$ for all $z \in \Omega$.

Theorem

(See [3]) Suppose that the domain Ω is an X - weak domain of holomorphy and that X satisfies the above assumptions. Then Ω is an X - domain of holomorphy and the set

$$S = \{f \in X : f \text{ is non - extendable}\}$$

is a G_δ and dense subset of X.

Remark

In [2] it has been proved that S contains a G_δ and dense subset of X. The new fact is that S is itself a G_δ set. This follows from Montel's theorem in Complex analysis. By using standard arguments, if Ω is a domain in a separable Banach space, the set S^c can be written as a denumerable union of sets of the form

$$L(V, M) = \Big\{ f \in X : \text{ there exists } F \text{ holomorphic in } b_2$$

$$\text{such that } |F(z)| \leq M \text{ for all } z \in b_2 \text{ and } f_{|b_1} = F_{|b_1} \Big\}$$

where $V = (b_1, b_2)$.

It suffices to prove that $L(V, M)$ is closed in X. If $\{f_n\}_{n\geq 1}$ is a sequence in $L(V, M)$ converging in the topology of X towards a function $f \in X$, then we will show that f belongs in $L(V, M)$. For every $n \geq 1$ there exist a holomorphic function F_n on b_2 bounded by M such that $F_{n|b_1} = f_{n|b_1}$. By Montel's theorem there exist a subsequence $\{F_{k_n}\}_{n\geq 1}$ converging uniformly on the compact subsets of b_2 towards a holomorphic function F bounded by M on b_2. Thus, $F_{k_n}(z) \to F(z)$ for every $z \in b_1$, because $b_1 \subseteq b_2$.

By our assumptions on X, the convergence $f_n \to f$ in X implies the pointwise convergence $f_n(z) \to f(z)$ for all $z \in \Omega$; in particular for all $z \in b_1$. Thus, $F_{|b_1} = f_{|b_1}$, which implies that $f \in L(V, M)$ and we have proved that the set $L(V, M)$ is closed. Therefore, S^c is an F_σ and S is a G_δ in X.

For completeness, we show that $L(V,M)$ has empty interior, under the assumption that Ω is an X - weak domain of holomorphy. Then Baire's theorem yields the result. Suppose that $f \in L(V,M)^o$ to arrive at a contradiction. By assumption, there exists a function $f_V \in X$ such that $f_{V|b_1}$ does not admit any bounded holomorphic extension in b_2. Then $f + \frac{1}{n} f_V \to f \in X$ as $n \to +\infty$, which implies that $f + \frac{1}{n_0} f_V \in L(V,M)$, a some natural number n_0. Thus, there exist two holomorphic functions F and G bounded by M on b_2 extending $f_{|b_1}$ and $f + \frac{1}{n_0} f_{V|b_1}$ respectively. It follows that the function $n_0(G-F)$ is a bounded holomorphic extension of $f_{V|b_1}$ on b_2 . This contradicts our assumption and the proof is complete.

Similar results hold for functions totally unbounded. Let f be a holomorphic function on the domain Ω. Consider any point $\zeta \in \partial\Omega$ and any ball $B(\zeta, r), r > 0$. Let A be any component of the intersection $\Omega \cap B(\zeta, r)$. We say that f is totally unbounded if $f_{|A}$ is unbounded for every choice of $\zeta \in \partial\Omega, r > 0$ and A as previously. It follows that if f is totally unbounded, then f is non - extendable. Also, the set of totally unbounded functions in X is either void or G_δ and dense ([4], [1]).

A first research direction is to investigate whether $S \cup \{0\}$ contains a dense vector subspace (algebraic genericity) or an infinite dimensional closed vector subspace (spaceability). In [2] such results are presented, mainly in one complex variable, and R. Aron has contributed in this direction. I believe that it is possible to add new such results in one, several or infinitely many complex variables. We can have functions in X whose restrictions to many complex lines are non - extendable or even totally unbounded and hence non - extendable from Ω. In some domains Ω Globevnik and Stout obtained this result for all complex lines ([3]). Is this generic or spaceable? Does this extends to infinitely many variables?

Most of the spaces $X = X(\Omega)$ are defined by requiring that a property holds when we approach the whole boundary of Ω. For instance, the space $A^p(\Omega)$ consists of all holomorphic functions f on Ω whose $p+1$ first derivatives extend continuously to the whole boundary $\partial\Omega$. These spaces can be generalised by requiring that a property holds when we approach only a part of the boundary and not necessarily the whole boundary and this in one, several or infinitely many variables. What is the natural topology on these spaces, are they complete metric spaces, so that Baire's theorem can be applied, and do the previous results, as well as other ones, extend to these spaces? Such spaces can be defined in analogy to the spaces A^p, H^∞_p, Bergman Spaces, Hardy Spaces, Bloch, BMOA, Nevanlinna class and other.

Furthermore, we can consider combinations (intersections) of such spaces, by requiring several properties to hold when we approach several parts of the boundary respectively. These parts of the boundary may be finitely many or infinitely denumerably many, they may intersect each other or not and they can cover the whole boundary or not. Endowing these spaces with their natural topology they will become Fréchet spaces and the previous results can be extended to these spaces.

Generic nowhere differentiable functions on a part of $\partial\Omega$ can be found in the generalization of the space $A(\Omega) = A^0(\Omega)$, as well as in the generalization of $A^p(\Omega)$, where the p - th derivative of f will be nowhere differentiable on a part of $\partial\Omega$. Moreover, this will be generic. Is it true that the new spaces $A^p(\Omega)$ relative to a part of $\partial\Omega$ is included in the space C^p relative to a part of $\partial\Omega$, where C^p is defined by differentiation with respect to the position and not with respect to a parametrization of this part of the boundary? This allows one to consider domains Ω where $\partial\Omega$ is not a curve and does not have a parametrization. In fact, is there an equality $A^p(\Omega, J) = A^0(\Omega, J) \cap C^p(J)$?

Furthermore, consider a Jordan domain Ω with the property that its boundary contains a rectifiable arc and let ϕ be a Riemann map from the open unit disc $\mathbb{D}$ onto Ω. Then, the derivative ϕ' will belong to the generalization of the Hardy H^1 when we approach the corresponding arc of the unit circle $\partial\mathbb{D}$. What are the properties of the functions belonging to these new Hardy spaces? They have tangential limits almost everywhere on the corresponding arc. What can be said about their zeros? Are there extensions when the disc $\mathbb{D}$ is replaced by other domains in one, several or infinitely many variables? Is the Picard property valid generically in the new Hardy spaces close to every point of the chosen part of the boundary? Is there something analogous for the generalization of Bergman spaces? What are the possible generalizations of Dirichlet spaces?

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