Weakly compactly generated Banach spaces and some of their relatives classified by using projectional skeletons

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Joint work with M. Fabian, Praha

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Supported in part by MICINN MTM 2014-57838-C2-2-P (Spain)

NonLinear Functional Analysis
October 17-20, 2017
1. Weakly compactly generated spaces
2. Projections
3. Projectional Resolutions of the Identity
4. Projectional Skeletons
5. Projectional Generators
6. WLD spaces
7. WCG spaces
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- $L_1(\mu)$ for $\mu$-finite.
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**How to build a projection?**

**Lemma**

For a Banach space $X$, closed linear subspaces $V \subset X$, $Y \subset X^*$, the following conditions are equivalent (TFAE):

1. $X = V \oplus Y^\perp$, and $P : X \to V$ has norm $\leq r$.
2. $V$ separates points of $Y$.
3. $V^\perp \cap Y$ is the $r$-norm closure of $V^\perp$.

We say that the couple $(V, Y)$ gives an $r$-bounded projection $P$.
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(i) $X = V \oplus Y_\perp$, $P : X \to V$ has norm $\leq r$.

(ii) $V$ separates points of $Y$, $W^* (V_\perp \cap Y_w^*) = \{0\}$, and $B_Y r$-norms $V (\|v\| \leq r \sup \langle v, B_Y \rangle, v \in V)$.

We say that the couple $(V, Y)$ gives an $r$-bounded projection $P_V \times Y$ (on $V$).
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$\{P_\alpha : \omega_0 \leq \alpha \leq \mu\}$ projections.

(i) $P_{\omega_0} = 0$, $P_{\mu} = Id_X$.

(ii) $\|P_\alpha\| = 1$, $\forall \alpha \in (\omega_0, \mu]$.

(iii) $dens(P_\alpha(X)) \leq |\alpha|$, $\forall \alpha \in [\omega_0, \mu]$.

(iv) $P_\alpha P_\beta = P_{\min\{\alpha, \beta\}}$, $\forall \alpha \in [\omega_0, \mu]$.

(v) For $x \in X$, $\alpha \mapsto P_\alpha(x)$ continuous.
Amir–Lindenstrauss’68: Building a \textbf{Projectional Resolution of the Identity}.

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V. Montesinos WCG by means of skeletons
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Assume $X$ has a projectional skeleton. Then:

- $X$ admits an $r$-PRI for some $r$.
- $X$ linearly and continuously injects into $c_0(\text{dens } X)$.
- $X$ has a Markushevich basis, (thus $\ell_\infty$ does not admit a PS);
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A Projectional Generator (PG) (Orihuela, Valdivia, and precursors John, Zizler, Vašák, Gul’ko, Fabian, Plichko)
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Theorem (Orihuela, Valdivia)

\((N, \Phi) \overset{PG}{\Rightarrow} PRI.\)
Natural examples of PG’s

- \((WCG)\) \(K \subset X\) w-K that generates \(X\).
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- (WCG) \( K \subset X \) \( w \)-K that generates \( X \). Put \( \Phi(x^*) \in K \) where \( x^* \) attains the maximum \( \Rightarrow (X^*, \Phi) \) PG.
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- (WCG) $K \subset X$ $w$-K that generates $X$. Put $\Phi(x^*) \in K$ where $x^*$ attains the maximum $\Rightarrow (X^*, \Phi)$ PG.

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- **(WCG, SWCG, WCD, WLD, 1-Pličko)** $X$ Banach, $M \subset X$ linearly dense, $N \subset X^*$ 1-norming, $\forall x^* \in N$, supp$_Mx^*$ countable.
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WCG by means of skeletons
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  This is exactly the class of 1-Pličko spaces.

  Particular case: $X$ with M-basis st $\forall x^* \in X^*$, supp $Mx^*$ is countable.

  This class is exactly WLD, i.e., $X$ with $(B_{X^*}, w^*)$ Corson.
Recall the definition of a PS:

\[ \Gamma : \gamma \in \Gamma \text{ projections } X \rightarrow X, \text{ where } \gamma \in \Gamma \text{ partially ordered, directed upwards, } \sigma\text{-complete}, \]

(i) \[ \mathcal{P}_\gamma X \text{ separable}, \forall \gamma \in \Gamma. \]

(ii) \[ X = \bigcup_{\gamma \in \Gamma} \mathcal{P}_\gamma X. \]

(iii) \[ \mathcal{P}_\gamma \circ \mathcal{P}_\beta = \mathcal{P}_\beta \circ \mathcal{P}_\gamma = \mathcal{P}_\gamma \text{ if } \gamma \leq \beta, \]

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Problem Characterize \( X \) having PS.

\[ \mathcal{S}(X) \text{ separable subspaces of } X, \mathcal{S} \subseteq \subseteq (X \times X^*) := \{ V \times Y : V \in \mathcal{S}(X), Y \in \mathcal{S}(X^*) \} \text{ (rectangles).} \]

\[ \mathcal{R} \subseteq \subseteq (X, X^*) \text{ rich subfamily (directed upwards, cofinal, } \sigma\text{-complete).} \]
Recall the definition of a **PS**:
\[ \{ P_\gamma : \gamma \in \Gamma \} \] projections \( X \to X \), where \( \gamma \in \Gamma \) partially ordered, directed upwards, \( \sigma \)-complete,
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\( S(X) \) separable subspaces of \( X \),
\( S_{\square}(X \times X^*) := \{ V \times Y : V \in S(X), Y \in S(X^*) \} \) (rectangles).
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\( \mathcal{R} \subset S_\square(X, X^*) \) rich subfamily (directed upwards, cofinal, \( \sigma\)-complete).
Theorem

$X$ with a PG $(D, \phi)$, $D \subset X^*$ r-norming. Then $\exists \mathcal{R}$ rich in $S_\Box(X \times X^*)$ giving r-bounded projections.
Theorem

\( X \) with a PG \((D, \phi)\), \( D \subset X^*\) r-norming. Then \( \exists \mathcal{R} \) rich in \( S_\square(X \times X^*) \) giving r-bounded projections.

Theorem

\( D \subset X^*\) r-norming, \( \mathcal{R} \) rich in \( S_\square(X \times X^*) \), giving r-bounded projections. Then \( \{ P_\gamma : \gamma \in \mathcal{R} \} \) PS with \( D \subset \bigcup_{\mathcal{R}} P_\gamma^* X^* \).
Theorem

$X$ with a PG $(D, \phi)$, $D \subset X^*$ $r$-norming. Then $\exists \mathcal{R}$ rich in $S_{\square}(X \times X^*)$ giving $r$-bounded projections.

Theorem

$D \subset X^*$ $r$-norming, $\mathcal{R}$ rich in $S_{\square}(X \times X^*)$, giving $r$-bounded projections. Then $\{P_\gamma : \gamma \in \mathcal{R}\}$ PS with $D \subset \bigcup_{\mathcal{R}} P^*_\gamma X^*$.

Corollary

$X$ with $r$-PG. Then $X$ has an $r$-PS.
Theorem

$X$ Banach, $M \subset X$ linearly dense, $D \subset X^*$ st

- $\forall x^* \in D$, $\Phi(x^*) := \text{supp}_M x^*$ countable.
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$X$ Banach, $M \subset X$ linearly dense, $D \subset X^*$ st

$\forall x^* \in D$, $\Phi(x^*) := \text{supp}_M x^*$ countable. Then

$R := \{ V \times Y \in S_\square (X \times D) : M \setminus V \subset Y_\perp \}$ is rich in $S_\square (X \times D)$. 
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• If moreover $D$ is r-norming, $\Gamma$ rich in $S_{\square}(X \times D)$ given by $(D, \Phi)$. 
Theorem

$X$ with a PG $(D, \phi)$, $D \subset X^*$ r-norming. Then $\exists \mathcal{R}$ rich in $S_\square(X \times X^*)$ giving $r$-bounded projections.

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• $\forall x^* \in D$, $\Phi(x^*) := \text{supp}_M x^*$ countable. Then
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• If moreover $D$ is r-norming, $\Gamma$ rich in $S_\square(X \times D)$ given by $(D, \Phi)$.
**Theorem**

\[ X \text{ with a PG } (D, \phi), \ D \subset X^* \text{ r-norming. Then } \exists \ R \text{ rich in } S(\Box(X \times X^*)) \text{ giving } r\text{-bounded projections.} \]

**Theorem**

\[ X \text{ Banach, } M \subset X \text{ linearly dense, } D \subset X^* \text{ st} \]

- \( \forall x^* \in D, \Phi(x^*) := \text{supp}_M x^* \text{ countable. Then} \)

\[ R := \{ V \times Y \in S(\Box(X \times D)) : M \setminus V \subset Y_\perp \} \text{ is rich in } S(\Box(X \times D)). \]

- **If moreover** \( D \) **is r-norming,** \( \Gamma \) **rich in** \( S(\Box(X \times D)) \) **given by** \( (D, \Phi) \). **Then** \( \text{PS generated by } R \cap \Gamma \text{ is commutative.} \)
$X$ WCG, then $(B_{X^*}, w^*)$ is an Eberlein compact.
$X$ WCG, then $(B_{X^*}, w^*)$ is an Eberlein compact. Even more, is a compact in a $\Sigma$-product (i.e., a Corson compact).
X WCG, then $(B_{X^*}, w^*)$ is an Eberlein compact. Even more, is a compact in a $\Sigma$-product (i.e., a Corson compact). This property characterizes the WLD spaces.
**Theorem**

\[ X \text{ WLD} \iff \exists \text{ commutative PS.} \]
\{ P_\gamma : \gamma \in \Gamma \} \text{ PS. Then } P_{\gamma_n}^* x^* \overset{w^*}{\rightarrow} P_\gamma x^* \text{ whenever } \gamma_n \uparrow \gamma, \forall x^* \in X^*.
{P_\gamma : \gamma \in \Gamma} \text{ PS. Then } P_{\gamma_n}^* x^* \xrightarrow{w^*} P_\gamma x^* \text{ whenever } \gamma_n \nearrow \gamma, \\
\forall x^* \in X^*.
\text{A } \subset X \text{ bounded, } \varepsilon \geq 0,
\{ P_\gamma : \gamma \in \Gamma \} \text{ PS. Then } P_{\gamma_n}^* x^* \xrightarrow{w^*} P_\gamma x^* \text{ whenever } \gamma_n \nearrow \gamma, \\
\forall x^* \in X^*.

A \subset X \text{ bounded, } \varepsilon \geq 0, \text{ a PS } (P_\gamma : \gamma \in \Gamma) \text{ in } X \text{ is } A-\varepsilon\text{-shrinking if } \forall x^* \in X^*,

\text{if } \varepsilon = 0, \text{ A-shrinking.}

Theorem (Fabian–M.)

TFAE:

(i) $X$ WCG.

(ii) $\exists A \subset X$ closed abs.convex, bounded, lin. dense, and a PS $P_\gamma : \gamma \in \Gamma$ in $X$ is $A-\varepsilon$-shrinking.
Characterizing WCG by skeletons

\{P_{\gamma} : \gamma \in \Gamma\} \text{ PS. Then } P_{\gamma_n}^* x^* \xrightarrow{w^*} P_{\gamma} x^* \text{ whenever } \gamma_n \uparrow \gamma, \forall x^* \in X^*.

A \subset X \text{ bounded, } \varepsilon \geq 0, \text{ a PS } (P_{\gamma} : \gamma \in \Gamma) \text{ in } X \text{ is } A-\varepsilon\text{-shrinking if } \forall x^* \in X^*,

\limsup_{j \to \infty} \rho_A(P_{\gamma_j}^* x^*, P_{\sup_{\gamma_i}}^* x^*) \leq \varepsilon \|x^*\|;

whenever \gamma_n \uparrow \gamma.
\{P_\gamma : \gamma \in \Gamma\} \text{ PS. Then } P_{\gamma_n}^* x^* \overset{w^*}{\to} P_\gamma x^* \text{ whenever } \gamma_n \uparrow \gamma, \forall x^* \in X^*.

If } A \subset X \text{ bounded, } \varepsilon \geq 0, \text{ a PS } (P_\gamma : \gamma \in \Gamma) \text{ in } X \text{ is } A-\varepsilon-\text{shrinking if } \forall x^* \in X^*,

\limsup_{j \to \infty} \rho_A( P_{\gamma_j}^* x^*, P_{\sup \gamma_i}^* x^*) \leq \varepsilon \|x^*\|;

\text{ whenever } \gamma_n \uparrow \gamma.

If } \varepsilon = 0, \text{ } A\text{-shrinking.}
Characterizing WCG by skeletons

$\{P_\gamma : \gamma \in \Gamma\}$ PS. Then $P_{\gamma_n}^* x^* \overset{w^*}{\to} P_\gamma x^*$ whenever $\gamma_n \rightharpoonup \gamma$, $\forall x^* \in X^*$.

$A \subset X$ bounded, $\varepsilon \geq 0$, a PS $(P_\gamma : \gamma \in \Gamma)$ in $X$ is $A$-$\varepsilon$-shrinking if $\forall x^* \in X^*$,

$$\limsup_{j \to \infty} \rho_A(P_{\gamma_j}^* x^*, P_{\sup \gamma_i}^* x^*) \leq \varepsilon \|x^*\|;$$

whenever $\gamma_n \rightharpoonup \gamma$.

if $\varepsilon = 0$, $A$-shrinking.

**Theorem (Fabian–M.)**

TFAE:

(i) $X$ WCG.

V. Montesinos  WCG by means of skeletons
\{P_\gamma : \gamma \in \Gamma\} \text{ PS. Then } P_{\gamma_n}^* x^* \xrightarrow{w^*} P_\gamma x^* \text{ whenever } \gamma_n \uparrow \gamma, \\
\forall x^* \in X^*.

A \subset X \text{ bounded, } \varepsilon \geq 0, \text{ a PS } (P_\gamma : \gamma \in \Gamma) \text{ in } X \text{ is } A-\varepsilon\text{-shrinking if } \forall x^* \in X^*,

\limsup_{j \to \infty} \rho_A(P_{\gamma_j}^* x^*, P_{\sup_{\gamma_i}}^* x^*) \leq \varepsilon \|x^*\|;

\text{ whenever } \gamma_n \uparrow \gamma.

\text{ if } \varepsilon = 0, \text{ A-shrinking.}

**Theorem (Fabian–M.)**

* TFAE: 
  (i) $X$ WCG.
  (ii) $\exists A \subset X$ closed abs.convex, bounded, lin. dense, and a PS $A$-shrinking, fixing $A$. 
$X \subset WCG$, (in short SWCG) not always WCG (Rosenthal).
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$X$ SWCG $\iff (B_{X^*}, w^*)$ Eberlein compact.
$X \subset \text{WCG}$, (in short \textbf{SWCG}) not always \text{WCG} (Rosenthal).

$X \text{ SWCG} \iff (B_{X^*}, w^*) \text{ Eberlein compact}$.

$C(K) \text{ SWCG} \iff C(K) \text{ WCG} \iff K \text{ Eberlein compact}$


\textbf{Theorem (Fabian–M.)}

TFAE:

(i) $X \text{ SWCG}$.

(ii) \exists PS, \exists \{A_n \subset B_X\} \text{ closed abs.convex, lin. dense}, \exists \{\varepsilon_n\} \text{ st } \forall n \in \mathbb{N}, \text{ the PS is } A_n\text{-}\varepsilon_n\text{-shrinking, fixes } A_n, \text{ and } \bigcup \varepsilon_n < \varepsilon \Rightarrow A_n = B_X$, \forall \varepsilon > 0.
$X \subset WCG$, (in short SWCG) not always WCG (Rosenthal).
$X$ SWCG $\iff (B_{X^*}, w^*)$ Eberlein compact.
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**Theorem (Fabian–M.)**

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(i) $X$ SWCG.
Characterizing SWCG by skeletons

$X \subset WCG$, (in short SWCG) not always WCG (Rosenthal).

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Characterizing SWCG by skeletons

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**Theorem (Fabian–M.)**

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**Theorem (Fabian–M.)**

**TFAE:**

(i) $X$ SWCG.

(ii) $\exists$ PS, $\exists \{A_n \subset B_X\}$ closed abs.convex, lin. dense, $\exists\{\epsilon_n\}$ st

$\forall n \in \mathbb{N}$, the PS is $A_n$-$\epsilon_n$-shrinking, fixes $A_n$, and $\bigcup_{\epsilon_n < \epsilon} A_n = B_X$, $\forall \epsilon > 0$.  

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Theorem (Fabian–M.–Zizler’2004)

$X$ Banach. TFAE:

(i) $X$ SWCG.

(ii) $\forall \varepsilon > 0, \exists X = \bigcup_{n=1}^{\infty} X_{\varepsilon n}$ is $\varepsilon$-w-compact, $\forall n \in \mathbb{N}$.
Theorem (Fabian–M.–Zizler’2004)

$X$ Banach. TFAE:
(i) $X$ SWCG.
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Theorem (Fabian–M.–Zizler’2004)

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An M-basis \((\Gamma, \Gamma')\) is **shrinking** if \(\Gamma^*\) is \(\| \cdot \|\)-total.
An M-basis \((\Gamma, \Gamma')\) is shrinking if \(\Gamma^*\) is \(\| \cdot \|\)-total.

**Theorem**

\[ X \text{ WCG Asplund} \iff \exists \text{ shrinking M-basis.} \]
an M-basis is $\sigma$-shrinking if $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ st $\forall U \in \mathcal{N}(0)$ in $(X^{**}, \| \cdot \|)$, $\forall \gamma \in \Gamma$, $\exists n \in \mathbb{N}$ st $\gamma \in \Gamma_n$, $\Gamma'_n \subset U$ (derivative in $(X^{**}, w^*)$).
an M-basis is \( \sigma \)-shrinking if \( \Gamma = \bigcup_{n=1}^{\infty} \Gamma_n \) st \( \forall U \in \mathcal{N}(0) \) in \((X^{**}, \| \cdot \|)\), \( \forall \gamma \in \Gamma \), \( \exists n \in \mathbb{N} \) st \( \gamma \in \Gamma_n \), \( \Gamma'_n \subset U \) (derivative in \((X^{**}, w^*)\)).

**Theorem (Fabian–M.–Zizler’2005)**

*TFAE*

(i) \( X \) SWCG.
an M-basis is \( \sigma \)-shrinking if \( \Gamma = \bigcup_{n=1}^{\infty} \Gamma_n \) st \( \forall U \in \mathcal{N}(0) \) in \((X^{**}, \| \cdot \|)\), \( \forall \gamma \in \Gamma \), \( \exists n \in \mathbb{N} \) st \( \gamma \in \Gamma_n \), \( \Gamma'_n \subset U \) (derivative in \((X^{**}, w^*)\)).

Theorem (Fabian–M.–Zizler’2005)

\( TFAE \)

(i) \( X \) SWCG.
(ii) \( \exists \sigma \)-shrinking \( M \)-basis.
an M-basis is \( \sigma \)-shrinking if \( \Gamma = \bigcup_{n=1}^{\infty} \Gamma_n \) st \( \forall U \in \mathcal{N}(0) \) in \((X^{**}, \| \cdot \|)\), \( \forall \gamma \in \Gamma \), \( \exists n \in \mathbb{N} \) st \( \gamma \in \Gamma_n \), \( \Gamma'_n \subset U \) (derivative in \((X^{**}, w^*)\)).

Theorem (Fabian–M.–Zizler'2005)

\textit{TFAE}

(i) \( X \) SWCG.

(ii) \( \exists \ \sigma \)-shrinking M-basis.

(iii) \((B_{X^*}, w^*)\) Eberlein compact.
an M-basis is $\sigma$-shrinking if $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ st $\forall U \in \mathcal{N}(0)$ in $(X^{**}, \| \cdot \|)$, $\forall \gamma \in \Gamma$, $\exists n \in \mathbb{N}$ st $\gamma \in \Gamma_n$, $\Gamma'_n \subset U$ (derivative in $(X^{**}, \omega^*)$).

**Theorem (Fabian–M.–Zizler’2005)**

**TFAE**

(i) $X$ SWCG.

(ii) $\exists \sigma$-shrinking M-basis.

(iii) $(B_{X^*}, \omega^*)$ Eberlein compact.

Moreover, every M-basis is $\sigma$-shrinking.
Three more open problems

Problem
Characterize $\mathcal{K}$-analytic Banach spaces by using skeletons.
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Problem
Characterize Vašak (i.e., weakly countably determined) spaces by using skeletons.
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Characterize $\mathcal{K}$-analytic Banach spaces by using skeletons.

Problem
Characterize Vašak (i.e., weakly countably determined) spaces by using skeletons.

Problem
Characterize Banach spaces simultaneously Asplund and 1-Pličko by using skeletons.


M. Fabian; V. Montesinos.  
*WCG spaces and their subspaces grasped by projectional skeletons.*  
To appear.