The minimal volume of simplices containing a convex body

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The basics

Notation

- A *convex body* $K \subset \mathbb{R}^n$ is a compact convex set with nonempty interior.

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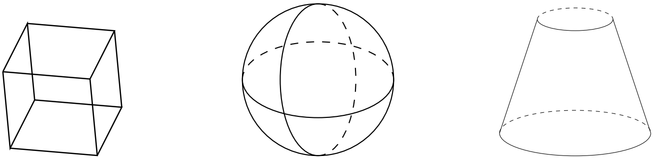
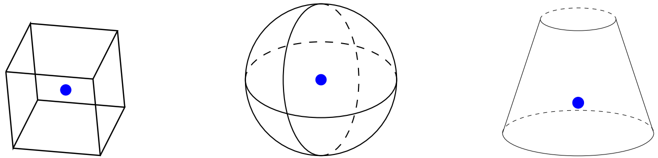


The basics

Notation

- A *convex body* $K \subset \mathbb{R}^n$ is a compact convex set with nonempty interior.
- The *barycenter* of K is $\mathrm{bar}(K) := \frac{1}{\mathrm{vol}(K)} \int_K x dx$.



The basics

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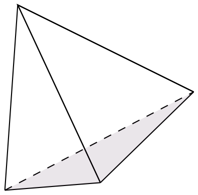
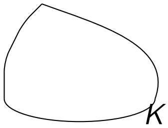
- A simplex in $\mathbb{R}^n$ is always an n-simplex, the convex hull of $n+1$ affinely independents points.



3-Simplex

The problem

Given a convex body K,approximate it a by a “simpler” one, in this case a simplex S.



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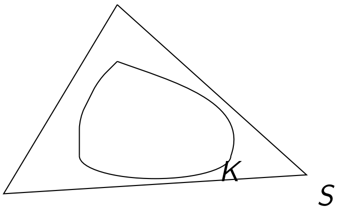

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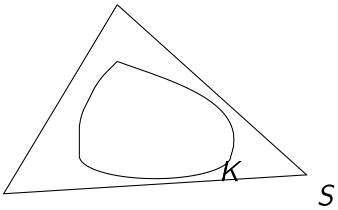

Approximating: enclosing it and having similar volume (Lebesgue measure).

Definition

For a convex body K we define the *outer simplex ratio* of K as

$$S_{\text{out}}(K) := \min \left(\frac{\text{vol}(S)}{\text{vol}(K)} \right)^{1/n},$$

the minimum is taken over all simplices $S \subset \mathbb{R}^n$ containing K.

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Problem

How small can $S_{\text{out}}(K)$ be?

Historical background

This is the generalization of an old geometrical problem.

$n = 2$

Wilhelm Gross at the early 20's: every convex body $K \subset \mathbb{R}^2$ can be inscribed in a triangle of area $2\mathrm{vol}(K)$.

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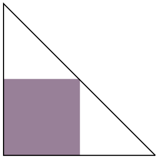
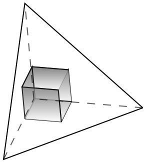
In this case the square is the worst posible fit.



$n = 3$

No exact bound has been given, $S_{\text{out}}(K) \leq \frac{9}{2}$ is conjectured. For a parallelepiped K a tethraedron of volume $\frac{9}{2}\text{vol}(K)$ can be found, but is not know if it is the best posible bound.



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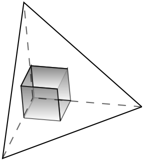
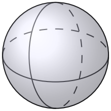


For $n \geq 4$ no exact bound was even conjectured.

Some examples

The Euclidean ball, B_2^n



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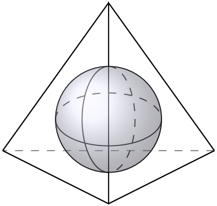
The Euclidean ball, B_2^n

$B_2^n \subset S$,the regular simplex, and is the smallest one.



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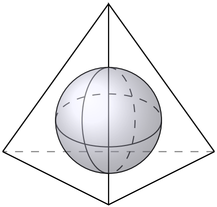

- $\mathrm{vol}(B_2^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$
- $\mathrm{vol}(S) = \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n}{2}}}{n!}$

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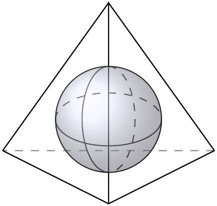

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Applying Stirling's formula:

$$S_{\mathrm{out}}(B_2^n) = \left(\frac{\mathrm{vol}(S)}{\mathrm{vol}(B_2^n)}\right)^{\frac{1}{n}} \approx n^{\frac{1}{2}}$$

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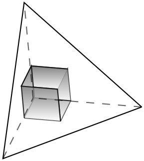

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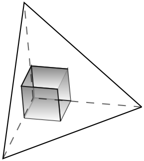

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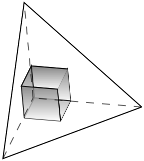

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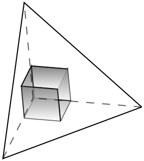

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For higher dimensions this is no longer the worst fit.

Known general bounds

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Up to an absolute constant this bound cannot be lessened, as $S_{\text{out}}(B_2^n) \approx \sqrt{n}$.

Polar bodies

Definition

Given a convex body $K \subset \mathbb{R}^n$, with $0 \in \text{int}(K)$, its polar body is

$$K^\circ = \{y \in \mathbb{R}^n \colon \langle x, y \rangle \leq 1 \, \forall x \in K\}.$$

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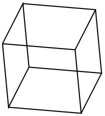

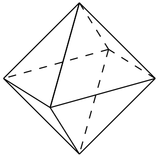

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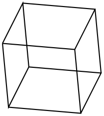

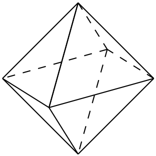

Some facts

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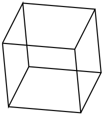

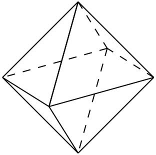

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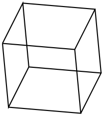

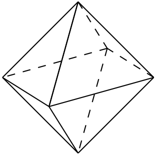

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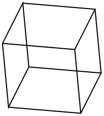

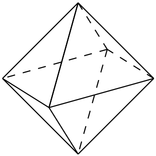

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- The polar of a simplex is also a simplex.
- If $0 \notin \text{int}(K)$, K° is unbounded.
- $(K^\circ)^\circ = K$ and $K \subset L \leftrightarrow \mathrm{L}^\circ \subset K^\circ$.

Polarity and volume

Theorem (Balschke-Santaló and Bourgain-Milman)

If K has barycenter at the origin then

$$(vol(K)vol(K^{\circ}))^{\frac{1}{n}} \approx vol(B_2^n)^{\frac{2}{n}} \approx \frac{1}{n}.$$

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- In this case polarity reverses volume.

A dual problem

Two equivalent problems

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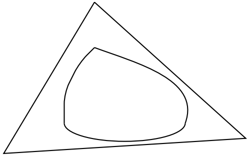
Original problem

Approximating a convex body by a simplex enclosing it of small volume



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Dual problem

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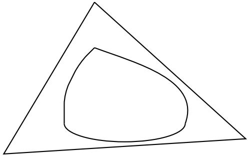

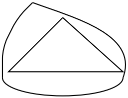

A dual problem

Two equivalent problems

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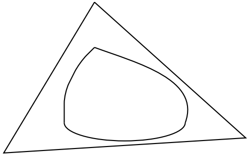
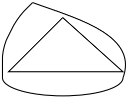
Approximating a convex body by a simplex enclosing it of small volume and same barycenter.

Dual problem

Approximating a convex body by a simplex inside it of large volume and same barycenter.





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Theorem (dual version)

$S_{inn}(K) \geq \frac{C}{\sqrt{n}}$ for some absolute constant $C > 0$.

Goal

Given $K \subset \mathbb{R}^n$ find a simplex $S \subset K$ satisfying:

1. $\mathrm{vol}(S)^{\frac{1}{n}} \geq C \frac{\mathrm{vol}(K)^{\frac{1}{n}}}{\sqrt{n}}$
2. $\mathrm{bar}(S) = \mathrm{bar}(K)$

with the same barycenter and large volume.

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A simplex inside K with same barycenter and large volume.

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- Prove that with postive probability there is a simplex similar to the objective.

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- Prove that with postive probability there is a simplex similar to the objective.
- Adjust it.

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Affine invariance

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- $S_{\text{inn}}(K)$ is a property of the affine class of K rather than a property of K.
- Choose a representative with a controllable uniform measure.

Isotropic bodies

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A convex body K is *isotropic* if:

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A convex body K is *isotropic* if:

- $vol(K) = 1$,
- $\mathrm{bar}(K) = 0$,
- there is $L_K \in \mathbb{R}$ such that $\int_K x_i x_j = L_K^2 \delta_{ij}$.
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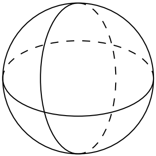

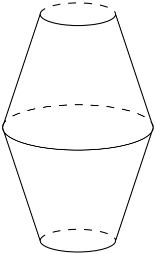

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Fact

Every convex body K has a unique (up to orthogonal transformations) affine isotropic image.

Random simplex on isotropic bodies

Proposition

Let K be an isotropic convex body and $X_1, \ldots X_n$ independent random variables distributed uniformly on K.
$S = co\{0, X_1, \ldots, X_n\}$.
With probabilty grater than $1 - e^{-n}$:

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- $\|bar(S)\| = \|\frac{1}{n+1} \sum_{i=1}^{n} X_i\| \leq L_K C_1$

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- $vol(S) = \frac{\det |X_1 \ldots X_n|}{n!} \geq \frac{C_2^n}{n^{\frac{n}{2}}} L_K^n$

Random simplex on isotropic bodies

Proposition

Let K be an isotropic convex body and $X_1, \dots X_n$ independent random variables distributed uniformly on K.
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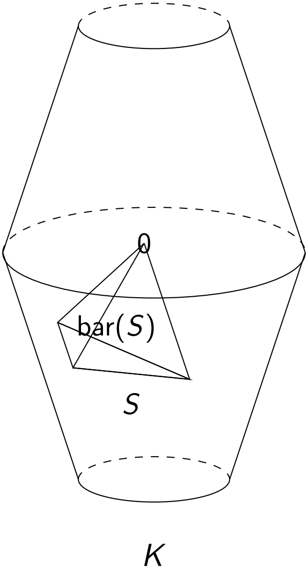
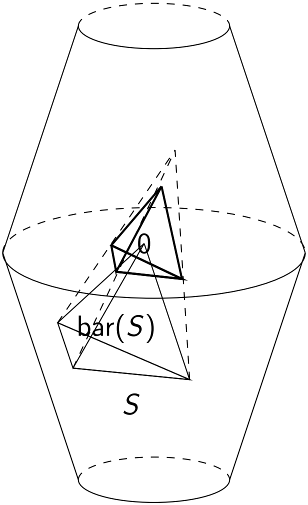
S has large volume and barycenter close to the origin.


0
bar(S)
S
K



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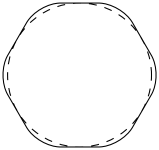
Case K symmetric



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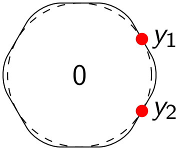

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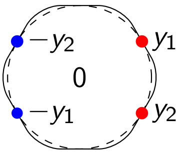

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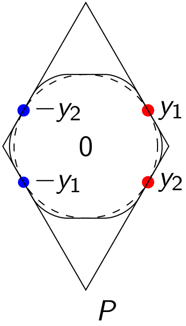

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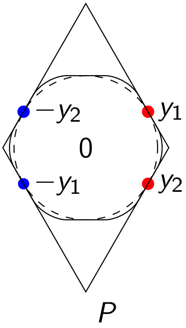

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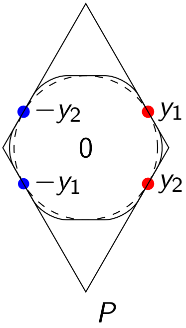

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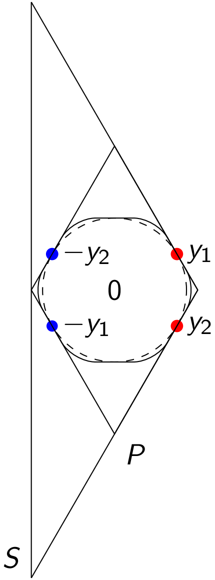

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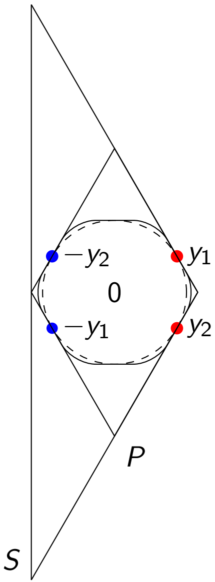

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General case

Rogers-Shephard Inequality

Given a convex body K with barycenter at the origin,

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A centered body can be approximated by outside by a symmetric body with essentialy the same volume.

$$\left(\frac{\mathrm{vol}(K - K)}{\mathrm{vol}(K)}\right)^{\frac{1}{n}} \leq 4.$$

Thank you!