

Analytic families of multilinear operators

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Introduction

A bounded measurable function σ on $\mathbb{R}^n \times \mathbb{R}^n$ (called a multiplier) leads to a bilinear operator W_σ defined by

$$W_\sigma(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle x, \xi + \eta \rangle} d\xi d\eta$$

for every f, g in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbb{R}^n and \widehat{f} is its **Fourier transform** of f defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \quad \xi \in \mathbb{R}^n.$$

- The study of such bilinear multiplier operators was initiated by Coifman and Meyer (1978). They proved that if $1 < p, q < \infty$, $1/r = 1/p + 1/q$ and σ satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq C_{\alpha, \beta} (|\xi| + |\eta|)^{-|\alpha| - |\beta|}$$

for sufficiently large multi-indices α and β , then W_σ extends to a bilinear operator from $L_p(\mathbb{R}^n) \times L_q(\mathbb{R}^n)$ into $L_{r, \infty}(\mathbb{R}^n)$ whenever $r > 1$.

- This result was later extended to the range $1 > r > 1/2$ by Grafakos and Torres (1996) and Kenig and Stein (1999).
- Multipliers that satisfy the Marcinkiewicz condition were studied by Grafakos and Kalton (2001).
- The first significant boundedness results concerning non-smooth symbols were proved by Lacey and Thiele (1997, 1999) who established that W_σ with $\sigma(\xi, \eta) = \text{sign}(\xi + \alpha\eta)$, $\alpha \in \mathbb{R} \setminus \{0, 1\}$ has a bounded extension from $L_p(\mathbb{R}^n) \times L_q(\mathbb{R}^n)$ to $L_r(\mathbb{R}^n)$ if $2/3 < r < \infty$, $1 < p, q \leq \infty$, and $1/r = 1/p + 1/q$.
- The bilinear Hilbert transform H_θ is defined for a parameter $\theta \in \mathbb{R}$ by

$$H_\theta(f, g)(x) := \lim_{\varepsilon \rightarrow 0} \int_{|t| > \varepsilon} f(x - t)g(x + \theta t) \frac{1}{t} dt, \quad x \in \mathbb{R}$$

for functions f, g from the Schwartz class $\mathcal{S}(\mathbb{R})$. The family $\{H_\theta\}$ was introduced by Calderón in his study of the first commutator, an operator arising in a series decomposition of the Cauchy integral along Lipschitz curves. In 1977 Calderón posed the question whether H_θ satisfies any L_p estimates.

- In their fundamental work (Ann. of Math. **149** (1999), 475-496) Lacey and Thiele proved that if $\theta \neq -1$, then the bilinear Hilbert transform H_θ extends to a bilinear operator from $L_p \times L_q$ into L_r whenever $1 < p, q \leq \infty$ and $1/p + 1/q = 1/r < 3/2$.
- The bilinear Hilbert transforms arise in a variety of other related known problems in bilinear Fourier analysis, e.g., in the study of the convergence of the mixed Fourier series of the form

$$\lim_{N \rightarrow \infty} \sum_{\substack{|m-\theta n| \leq N \\ |m-n| \leq N}} \hat{f}(m) \hat{g}(n) e^{2\pi i(m+n)x}$$

for functions f, g defined on $[0, 2\pi]$.

- Fan and Sato in 2001 were able to show the boundedness of the bilinear Hilbert transform H on the torus \mathbb{T}

$$H(f, g)(x) := \int_{\mathbb{T}} f(x-t)g(x+t) \operatorname{ctg}(\pi t) dt, \quad x \in \mathbb{T}$$

by transferring the result from \mathbb{R} . Their proof relies upon some DeLeeuw (1969) type transference methods for multilinear multipliers.

- Stein's interpolation theorem [Trans. Amer. Math. Soc. (1956)] for analytic families of operators between L^p spaces ($p \geq 1$) has found several significant applications in harmonic analysis. This theorem provides a generalization of the classical single-operator **Riesz-Thorin interpolation theorem** to a family $\{T_z\}$ of operators that depend analytically on a complex variable z .
- In the framework of Banach spaces, interpolation for analytic families of multilinear operators can be obtained via duality in a way similar to that used in the linear case. For instance, one may adapt the proofs in Zygmund book and Berg and Löfstrom for a single multilinear operator to a family of multilinear operators. However, this duality-based approach is not applicable to **quasi-Banach spaces** since their topological dual spaces may be **trivial**.

The open strip $\{z; 0 < \operatorname{Re} z < 1\}$ in the complex plane is denoted by S , its closure by \bar{S} and its boundary by ∂S .

Definition Let $A(S)$ be the space of scalar-valued functions, analytic in S and continuous and bounded in \bar{S} . For a given couple (A_0, A_1) of quasi-Banach spaces and A another quasi-Banach space satisfying $A \subset A_0 \cap A_1$, we denote by $\mathcal{F}(A)$ the space of all functions $f: S \rightarrow A$ that can be written as finite sums of the form

$$f(z) = \sum_{k=1}^N \varphi_k(z) a_k, \quad z \in \bar{S},$$

where $a_k \in A$ and $\varphi_k \in A(S)$. For every $f \in \mathcal{F}(A)$ we set

$$\|f\|_{\mathcal{F}(A)} = \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{A_0}, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{A_1} \right\}.$$

Definition A quasi-Banach couple is said to be **admissible** whenever $\|\cdot\|_\theta$ is a quasi-norm on $A_0 \cap A_1$, and in this case, the quasi-normed space $(A_0 \cap A_1, \|\cdot\|_\theta)$ is denoted by $(A_0, A_1)_\theta$.

Remark If A is dense in $A_0 \cap A_1$, then for every $a \in A$ we have

$$\|a\|_\theta = \inf\{\|f\|_{\mathcal{F}(A)}; f \in \mathcal{F}(A), f(\theta) = a\}.$$

Definition If there is a completion of $(A_0, A_1)_\theta$ which is set-theoretically contained in $A_0 + A_1$, then it is denoted by $[A_0, A_1]_\theta$.

Theorem [A. P. Calderón, *Studia Math.* (1964)] If (A_0, A_1) is a Banach couple, then for every $f \in \mathcal{F}(A_0 \cap A_1)$, and $0 < \theta < 1$,

$$\log \|f(\theta)\|_\theta \leq \int_{-\infty}^{\infty} \log \|f(it)\|_{A_0} P_0(\theta, t) dt + \int_{-\infty}^{\infty} \log \|f(1+it)\|_{A_1} P_1(\theta, t) dt,$$

where P_0 and P_1 are the Poisson kernels for the strip defined for $j \in \{0, 1\}$ by

$$P_j(x + iy, t) = \frac{e^{-\pi(t-y)} \sin \pi x}{\sin^2 \pi x + (\cos \pi x - (-1)^j e^{-\pi(t-y)})^2}, \quad x + iy \in \bar{S}.$$

Remark The same estimate holds in the case of quasi-Banach spaces; the proof is similar as in the Banach case.

Using the fact that the Poisson kernels satisfy

$$\int_{\mathbb{R}} P_0(\theta, t) dt = 1 - \theta, \quad \int_{\mathbb{R}} P_1(\theta, t) dt = \theta$$

together with Calderón's inequality and Jensen's inequality, and the concavity of the logarithmic function, we obtain the following result:

Lemma Let (A_0, A_1) be a couple of complex quasi-Banach spaces. For every f in $\mathcal{F}(A_0 \cap A_1)$, $0 < p_0, p_1 < \infty$, and $0 < \theta < 1$ we have

$$\|f(\theta)\|_{\theta} \leq \left(\frac{1}{1-\theta} \int_{-\infty}^{\infty} \|f(it)\|_{A_0}^{p_0} P_0(\theta, t) dt \right)^{\frac{1-\theta}{p_0}} \left(\frac{1}{\theta} \int_{-\infty}^{\infty} \|f(1+it)\|_{A_1}^{p_1} P_1(\theta, t) dt \right)^{\frac{\theta}{p_1}}$$

Analytic families of multilinear operators

Definition A continuous function $F: \bar{S} \rightarrow \mathbb{C}$ which is analytic in S is said to be of **admissible growth** if there is $0 \leq \alpha < \pi$ such that

$$\sup_{z \in \bar{S}} \frac{\log |F(z)|}{e^{\alpha |\operatorname{Im} z|}} < \infty.$$

Lemma [I. I. Hirschman, J. Analyse Math. (1953)] If a function $F: \bar{S} \rightarrow \mathbb{C}$ is analytic in S , continuous on \bar{S} , and is of admissible growth, then

$$\log |F(\theta)| \leq \int_{-\infty}^{\infty} \log |F(it)| P_0(\theta, t) dt + \int_{-\infty}^{\infty} \log |F(1+it)| P_1(\theta, t) dt.$$

Definition Let (Ω, Σ, μ) be a measure space and let $\mathcal{X}_1, \dots, \mathcal{X}_m$ be linear spaces. The family $\{T_z\}_{z \in \bar{S}}$ of multilinear operators

$$T: \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \rightarrow \tilde{L}^0(\mu)$$

is said to be **analytic** if for every $(x_1, \dots, x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$ and for almost every $\omega \in \Omega$ the function

$$z \mapsto T_z(x_1, \dots, x_m)(\omega), \quad z \in \bar{S} \quad (*)$$

is analytic in S and continuous on \bar{S} . Additionally, if for $j = 0$ and $j = 1$ the function

$$(t, \omega) \mapsto T_{j+it}(x_1, \dots, x_m)(\omega), \quad (t, \omega) \in \mathbb{R} \times \Omega$$

is $(\mathcal{L} \times \Sigma)$ -measurable for every $(x_1, \dots, x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, and for almost every $\omega \in \Omega$ the function given by formula (*) is of admissible growth, then the family $\{T_z\}_{z \in \bar{S}}$ is said to be an **admissible analytic family**. Here \mathcal{L} is the σ -algebra of Lebesgue's measurable sets in \mathbb{R} .

- A quasi-Banach lattice X is said to be **maximal** whenever $0 \leq f_n \uparrow f$ a.e., $f_n \in X$, and $\sup_{n \geq 1} \|f_n\|_X < \infty$ implies that $f \in X$ and $\|f_n\|_X \rightarrow \|f\|_X$.
- A quasi-Banach lattice X is said to be **p -convex** ($0 < p < \infty$) if there exists a constant $C > 0$ such that for any $f_1, \dots, f_n \in X$ we have

$$\left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_X \leq C \left(\sum_{k=1}^n \|f_k\|_X^p \right)^{1/p}.$$

The optimal constant C in this inequality is called the **p -convexity constant** of X , and is denoted, by $M^{(p)}(X)$.

- A quasi-Banach lattice is said to have **nontrivial convexity** whenever it is p -convex for some $0 < p < \infty$.

- If X_0 and X_1 are quasi-Banach lattices on a given measure space (Ω, Σ, μ) and $0 < \theta < 1$, we define the quasi-Banach lattice $X_\theta = X_0^{1-\theta} X_1^\theta$ to be the space of all $f \in L^0(\mu)$ such that $|f| \leq |f_0|^{1-\theta} |f_1|^\theta$ μ -a.e. for some $f_i \in X_i$ ($i = 0, 1$) and equipped with the quasi-norm

$$\|f\|_{X_\theta} = \inf \{ \|f_0\|_{X_0}^{1-\theta} \|f_1\|_{X_1}^\theta; |f| \leq |f_0|^{1-\theta} |f_1|^\theta \text{ } \mu\text{-a.e.} \}.$$

Main results

Theorem For each $1 \leq i \leq m$, let $\bar{X}_i = (X_{0i}, X_{1i})$ be admissible couples of quasi-Banach spaces, and let (Y_0, Y_1) be a couple of maximal quasi-Banach lattices on a measure space (Ω, Σ, μ) such that each Y_j is p_j -convex for $j = 0, 1$. Assume that \mathcal{X}_i is a dense linear subspace of $X_{0i} \cap X_{1i}$ for each $1 \leq i \leq m$, and that $\{T_z\}_{z \in \bar{S}}$ is an admissible analytic family of multilinear operators $T_z: \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \rightarrow Y_0 \cap Y_1$. Suppose that for every $(x_1, \dots, x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $t \in \mathbb{R}$ and $j = 0, 1$,

$$\|T_{j+it}(x_1, \dots, x_m)\|_{Y_j} \leq K_j(t) \|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}$$

where K_j are Lebesgue measurable functions such that $K_j \in L^{p_j}(P_j(\theta, \cdot) dt)$ for all $\theta \in (0, 1)$. Then for all $(x_1, \dots, x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, all $s \in \mathbb{R}$, and all $0 < \theta < 1$ we have

$$\|T_{\theta+is}(x_1, \dots, x_m)\|_{Y_0^{1-\theta} Y_1^\theta} \leq (M^{(p_0)}(Y_0))^{1-\theta} (M^{(p_1)}(Y_1))^\theta K_\theta(s) \prod_{i=1}^m \|x_i\|_{(X_{0i}, X_{1i})_\theta},$$

where

$$\log K_\theta(s) = \int P_0(\theta, t) \log K_0(t+s) dt + \int P_1(\theta, t) \log K_1(t+s) dt$$

Lemma Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space (Ω, Σ, μ) such that X_0 is p_0 -convex and X_1 is p_1 -convex. Then for every $0 < \theta < 1$ we have

$$\|x\|_{X_0^{1-\theta} X_1^\theta} \leq (M^{(p_0)}(X_0))^{1-\theta} (M^{(p_1)}(X_1))^\theta \|x\|_{(X_0, X_1)_\theta}, \quad x \in X_0 \cap X_1.$$

In particular (X_0, X_1) is an admissible quasi-Banach couple.

Lemma Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure (Ω, Σ, μ) . If $x_j \in X_j$ are such that $|x_j|$ ($j = 0, 1$) are bounded above and their non-zero values have positive lower bounds, then

$$|x_0|^{1-\theta} |x_1|^\theta \in (X_0, X_1)_\theta$$

and

$$\| |x_0|^{1-\theta} |x_1|^\theta \|_{(X_0, X_1)_\theta} \leq \|x_0\|_{X_0}^{1-\theta} \|x_1\|_{X_1}^\theta.$$

Corollary Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space (Ω, Σ, μ) . If $x \in X_0 \cap X_1$ has an order continuous norm in $X_0^{1-\theta} X_1^\theta$, then for every $0 < \theta < 1$,

$$\|x\|_{(X_0, X_1)_\theta} \leq \|x\|_{X_0^{1-\theta} X_1^\theta}.$$

Theorem Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space with nontrivial lattice convexity constants. If the space $X_0^{1-\theta} X_1^\theta$ has order continuous quasi-norm, then

$$[X_0, X_1]_\theta = X_0^{1-\theta} X_1^\theta$$

up equivalences of norms (isometrically, provided that lattice convexity constants are equal to 1). In particular this holds if at least one of the spaces X_0 or X_1 is order continuous.

Theorem For each $1 \leq i \leq m$, let (X_{0i}, X_{1i}) be complex quasi-Banach function lattices and let Y_j be complex p_j -convex maximal quasi-Banach function lattices with p_j -convexity constants equal 1 for $j = 0, 1$. Suppose that either X_{0i} or X_{1i} is order continuous for each $1 \leq i \leq m$. Let T be a multilinear operator defined on $(X_{01} + X_{11}) \times \cdots \times (X_{0m} + X_{1m})$ and taking values in $Y_0 + Y_1$ such that

$$T: X_{i1} \times \cdots \times X_{im} \rightarrow Y_i$$

is bounded with quasi-norm M_i for $i = 0, 1$. Then for $0 < \theta < 1$,

$$T: (X_{01})^{1-\theta}(X_{11})^\theta \times \cdots \times (X_{0m})^{1-\theta}(X_{1m})^\theta \rightarrow Y_0^{1-\theta}Y_1^\theta$$

is bounded with the quasi-norm

$$\|T\| \leq M_0^{1-\theta} M_1^\theta.$$

As an application we obtain the following interpolation theorem for operators proved by Kalton (1990), which was applied to study a problem in uniqueness of structure in quasi-Banach lattices (Kalton's proof uses a deep theorem by Nikishin and the theory of Hardy H_p -spaces on the unit disc).

Theorem Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measures spaces. Let X_i , $i = 0, 1$, be complex p_i -convex quasi-Banach lattices on $(\Omega_1, \Sigma_1, \mu_1)$ and let Y_i be complex p_i -convex maximal quasi-Banach lattices on $(\Omega_2, \Sigma_2, \mu_2)$ with p_i -convexity constants equal 1. Suppose that either X_0 or X_1 is order continuous. Let $T: X_0 + X_1 \rightarrow L^0(\mu_2)$ be a continuous operator such that $T(X_0) \subset Y_0$ and $T(X_1) \subset Y_1$. Then for $0 < \theta < 1$,

$$T: X_0^{1-\theta} X_1^\theta \rightarrow Y_0^{1-\theta} Y_1^\theta$$

and

$$\|T\|_{X_0^{1-\theta} X_1^\theta \rightarrow Y_0^{1-\theta} Y_1^\theta} \leq \|T\|_{X_0 \rightarrow Y_0}^{1-\theta} \|T\|_{X_1 \rightarrow Y_1}^\theta.$$

Applications to Lorentz and Hardy spaces

The Lorentz space $L_{p,q} := L_{p,q}(\Omega)$ on a measure space (Ω, Σ, μ) , $0 < p < \infty, 0 < q \leq \infty$ consists of all $f \in L^0(\mu)$ such that

$$\|f\|_{L_{p,q}} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} & \text{if } 0 < q < \infty \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty. \end{cases}$$

- The decreasing rearrangement f^* of f with respect to μ is defined by

$$f^*(t) = \inf\{s > 0; \mu_f(s) \leq t\}, \quad t \geq 0,$$

where

$$\mu_f(s) = \mu(\{\omega \in \Omega; |f(\omega)| > s\}), \quad s > 0.$$

Lemma Let $0 < p_j, q_j < \infty$ and let L_{p_j, q_j} for $j = 0, 1$ be Lorentz spaces on an infinite nonatomic measure space (Ω, Σ, μ) . Then for $0 < \theta < 1$ the quasi-norm of

$$X_\theta := (L_{p_0, q_0})^{1-\theta} (L_{p_1, q_1})^\theta$$

is equivalent to that of $L_{p, q}$, where $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $1/q = (1 - \theta)/q_0 + \theta/q_1$. Moreover for all $f \in X_\theta$ we have

$$2^{-1/p} \|f\|_{L_{p, q}} \leq \|f\|_{X_\theta} \leq \frac{2^{1/p}}{(\log 2)^s} s^s (p_0^{1-\theta} p_1^\theta)^s \|f\|_{L_{p, q}},$$

where $s = 1$ whenever $1 < p_0, p_1 < \infty$ and $1 \leq q_0, q_1 < \infty$ and $s > \max\{1/p_0, 1/q_0, 1/p_1, 1/q_1\}$ otherwise.

Theorem Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces. For $1 \leq i \leq m$, fix $0 < q_0, q_1, q_{0i}, q_{1i} < \infty$, $0 < r_0, r_1, r_{0i}, r_{1i} \leq \infty$ and for $0 < \theta < 1$, define q, r, q_i, r_i by setting

$$\frac{1}{q_i} = \frac{1-\theta}{q_{0i}} + \frac{\theta}{q_{1i}}, \quad \frac{1}{r_i} = \frac{1-\theta}{r_{0i}} + \frac{\theta}{r_{1i}}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Assume that \mathcal{X}_i is a dense linear subspace of $L_{q_{0i}, r_{0i}}(\Omega_1) \cap L_{q_{1i}, r_{1i}}(\Omega_1)$ and that $\{T_z\}_{z \in \bar{S}}$ is an admissible analytic family of multilinear operators $T_z: \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \rightarrow L_{q_0, r_0}(\Omega_2) \cap L_{q_1, r_1}(\Omega_2)$. Suppose that for every $(h_1, \dots, h_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $t \in \mathbb{R}$ and $j = 0, 1$, we have

$$\|T_{j+it}(h_1, \dots, h_m)\|_{L_{q_j, r_j}(\Omega_2)} \leq K_j(t) \|h_1\|_{L_{q_{j1}, r_{j1}}(\Omega_1)} \cdots \|h_m\|_{L_{q_{jm}, r_{jm}}(\Omega_1)}$$

where K_j are Lebesgue measurable functions such that $K_j \in L^{p_j}(P_j(\theta, \cdot) dt)$ for all $\theta \in (0, 1)$, where p_j is chosen so that $0 < p_j < q_j$ and $p_j \leq r_j$ for each $j = 0, 1$.

Then for all $(f_1, \dots, f_m) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_m$, $0 < \theta < 1$, and $s \in \mathbb{R}$ we have

$$\|T_{\theta+is}(f_1, \dots, f_m)\|_{(L_{q_0, r_0}, L_{q_1, r_1})_\theta} \leq C_\theta K_\theta(s) \prod_{i=1}^m \|f_i\|_{(L_{q_0, r_0}, L_{q_1, r_1})_\theta},$$

where

$$C(\theta) := \left(\frac{q_0}{q_0 - p_0} \right)^{\frac{1-\theta}{p_0}} \left(\frac{q_1}{q_1 - p_1} \right)^{\frac{\theta}{p_1}},$$

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta, t) \log K_0(t+s) dt + \int_{\mathbb{R}} P_1(\theta, t) \log K_1(t+s) dt.$$

If in addition the measures spaces are infinite and nonatomic, then for all (f_1, \dots, f_m) in $\mathcal{X}_1 \times \dots \times \mathcal{X}_m$ and $s \in \mathbb{R}$, and $0 < \theta < 1$ we have

$$\|T_{\theta+is}(f_1, \dots, f_m)\|_{L_{q,r}(\Omega_2)} \leq C \left(\frac{q_0}{q_0 - p_0}\right)^{\frac{1-\theta}{p_0}} \left(\frac{q_1}{q_1 - p_1}\right)^{\frac{\theta}{p_1}} K_\theta(s) \prod_{i=1}^m \|f_i\|_{L_{q_i,r_i}(\Omega_1)},$$

where

$$C = 2^{\frac{1}{q} + \sum_{i=1}^m \frac{1}{q_i}} \left(\frac{u q_0^{1-\theta} q_1^\theta}{\log 2}\right)^u$$

with $u = 1$ if $1 < q_0, q_1 < \infty$ and $1 \leq r_0, r_1 \leq \infty$, while $u > \max\{1/q_0, 1/q_1, 1/r_0, 1/r_1\}$ otherwise.

Suppose that there is an operator \mathcal{M} defined on a linear subspace of $\tilde{L}^0(\Omega, \Sigma, \mu)$ taking values in $\tilde{L}^0(\Omega, \Sigma, \mu)$ such that:

- (a) For $j = 0$ and $j = 1$ the function $(t, x) \mapsto \mathcal{M}(h(j + it, \cdot))(x)$, $(t, x) \in \mathbb{R} \times \Omega$ is $\mathcal{L} \times \Sigma$ -measurable for any function $h: \partial S \times \Omega \rightarrow \mathbb{C}$ such that $\omega \mapsto h(j + it, \omega)$ is Σ -measurable for almost all $t \in \mathbb{R}$.
- (b) $\mathcal{M}(\lambda h)(\omega) = |\lambda| \mathcal{M}(h)(\omega)$ for all $\lambda \in \mathbb{C}$.
- (c) For every function h as in above there is an exceptional set $E_h \in \Sigma$ with $\mu(E_h) = 0$ such that for $j \in \{0, 1\}$

$$\mathcal{M}\left(\int_{-\infty}^{\infty} h(j + it, \cdot) P_j(\theta, t) dt\right)(\omega) \leq \int_{-\infty}^{\infty} \mathcal{M}(h(t, \cdot))(\omega) P_j(\theta, t) dt$$

for all $\theta \in (0, 1)$, and all $\omega \notin E_h$. Moreover, $E_{\psi h} = E_h$ for every analytic function ψ on S which is bounded on \bar{S} .

For each $1 \leq i \leq m$, let $\bar{X}_i = (X_{0i}, X_{1i})$ be admissible couples of quasi-Banach spaces, and let (Y_0, Y_1) be a couple of complex maximal quasi-Banach lattices on a measure space (Ω, Σ, μ) such that each Y_j is p_j -convex for $j = 0, 1$. Assume that \mathcal{X}_i is a dense linear subspace of $X_{0i} \cap X_{1i}$ for each $1 \leq i \leq m$, and that $\{T_z\}_{z \in \bar{S}}$ is an admissible analytic family of multilinear operators $T_z: \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \rightarrow Y_0 \cap Y_1$. Assume that \mathcal{M} is defined on the range of T_z , takes values in $L^0(\Omega, \Sigma, \mu)$, and satisfies conditions (a), (b) and (c).

Theorem Suppose that for every $(x_1, \dots, x_m) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_m$, $t \in \mathbb{R}$ and

$$\|\mathcal{M}(T_{j+it}(x_1, \dots, x_m))\|_{Y_j} \leq K_j(t) \|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}, \quad j = 0, 1,$$

where K_j are Lebesgue measurable functions such that $K_j \in L^{p_j}(P_j(\theta, \cdot) dt)$ for all $\theta \in (0, 1)$. Then for all $(x_1, \dots, x_m) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_m$, $s \in \mathbb{R}$, and $0 < \theta < 1$,

$$\|\mathcal{M}(T_{\theta+is}(x_1, \dots, x_m))\|_{Y_0^{1-\theta} Y_1^\theta} \leq C_\theta K_\theta(s) \prod_{i=1}^m \|x_i\|_{(X_{0i}, X_{1i})_\theta},$$

where

$$C_\theta = (M^{(p_0)}(Y_0))^{1-\theta} (M^{(p_1)}(Y_1))^\theta,$$

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta, t) \log K_0(t+s) dt + \int_{\mathbb{R}} P_1(\theta, t) \log K_1(t+s) dt.$$

- If ϕ is a $C^\infty(\mathbb{R}^n)$ function with a compact support, $\phi_\delta(x) = \delta^{-n}\phi(x/\delta)$ for all $x \in \mathbb{R}^n$.

$$\mathcal{M}h(x) := \sup_{\delta>0} |(\phi_\delta * h)(x)|, \quad x \in \mathbb{R}^n.$$

- $Y_0 = L^{p_0}$, $Y_1 = L^{p_1}$, in which case $Y_0^{1-\theta} Y_1^\theta = L^p$, $1/p = (1-\theta)/p_0 + \theta/p_1$.

Definition The classical Hardy space H^p of Fefferman and Stein is defined by

$$\|h\|_{H^p} := \|\mathcal{M}(h)\|_{L^p}.$$

Corollary If $\{T_z\}$ is an admissible analytic family is such that

$$\|T_{j+it}(x_1, \dots, x_m)\|_{H^{p_j}} \leq K_j(t) \|x_1\|_{X_{j_1}} \cdots \|x_m\|_{X_{j_m}}, \quad j = 0, 1,$$

then

$$\|T_{\theta+s}(x_1, \dots, x_m)\|_{H^p} \leq K_\theta(s) \prod_{i=1}^m \|x_i\|_{(X_{0i}, X_{1i})_\theta}$$

for $0 < p_0, p_1 < \infty$, $s \in \mathbb{R}$, and $0 < \theta < 1$. Analogous estimates hold for the Hardy-Lorentz spaces $H^{q,r}$ where estimates of the form

$$\|T_{j+it}(x_1, \dots, x_m)\|_{H^{q_j, r_j}} \leq K_j(t) \|x_1\|_{X_{j_1}} \cdots \|x_m\|_{X_{j_m}}$$

for admissible analytic families $\{T_z\}$ when $j = 0, 1$ imply

$$\|T_{\theta+is}(x_1, \dots, x_m)\|_{H^{q,r}} \leq C K_\theta(s) \prod_{i=1}^m \|x_i\|_{(X_{0i}, X_{1i})_\theta},$$

where $0 < p_j < q_j < \infty$, $p_j \leq r_j \leq \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$ while $u = 1$ if $1 < q_0, q_1 < \infty$ and $1 \leq r_0, r_1 \leq \infty$ and $u > \max\{1/q_0, 1/q_1, 1/r_0, 1/r_1\}$ otherwise.

An application to the bilinear Bochner-Riesz operators

Stein's motivation to study analytic families of operators might have been the study of the Bochner-Riesz operators

$$B^\delta(f)(x) := \int_{|\xi| \leq 1} (1 - |\xi|^2)^\delta \widehat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

in which the “smoothness” variable δ affects the degree p of integrability of $B^\delta(f)$ on $L^p(\mathbb{R}^n)$.

Remark Using interpolation for analytic families of operators, **Stein** showed that whenever $\delta > (n-1) \left| \frac{1}{p} - \frac{1}{2} \right|$, then

$$B^\delta: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$$

is bounded for every $1 \leq p \leq \infty$.

- The bilinear Bochner-Riesz operators are defined on $\mathcal{S} \times \mathcal{S}$ by

$$S^\delta(f, g)(x) := \iint_{|\xi|^2 + |\eta|^2 \leq 1} (1 - |\xi|^2 - |\eta|^2)^\delta \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle x, \xi + \eta \rangle} d\xi d\eta$$

for every $f, g \in \mathcal{S}$.

- The bilinear Bochner-Riesz means S^z is defined by

$$S^z(f, g)(x) = \int \int K_z(x - y_1, x - y_2) f(y_1) g(y_2) dy_1 dy_2,$$

where that the kernel of $K_{\delta+it}$ is given by

$$K_{\delta+it}(x_1, x_2) = \frac{\Gamma(\delta + 1 + it)}{\pi^{\delta+it}} \frac{J_{\delta+it+n}(2\pi|x|)}{|x|^{\delta+it+n}}, \quad x = (x_1, x_2).$$

If $\delta > n - 1/2$, then using known asymptotics for Bessel functions we have that this kernel satisfies an estimate of the form:

$$|K_{\delta+it}(x_1, x_2)| \leq \frac{C(n + \delta + it)}{(1 + |x|)^{\delta+n+1/2}},$$

where $C(n + \delta + it)$ is a constant that satisfies

$$C(n + \delta + it) \leq C_{n+\delta} e^{B|t|^2}$$

for some $B > 0$ and so we have

$$|K_{\delta+it}(x_1, x_2)| \leq C_{n+\delta} e^{B|t|^2} \frac{1}{(1 + |x_1|)^{n+\epsilon}} \frac{1}{(1 + |x_2|)^{n+\epsilon}},$$

with $\epsilon = \frac{1}{2}(\delta - n - 1/2)$. It follows that the bilinear operator $S^{\delta+it}$ is bounded by a product of two linear operators, each of which has a good integrable kernel. It follows that

$$S^{\delta+it} : L^1 \times L^1 \rightarrow L^{1/2}$$

with constant $K_1(t) \leq C'_{n+\delta} e^{B|t|^2}$ whenever $\delta > n - 1/2$.

Theorem Let $1 < p < 2$. For any $\lambda > (2n - 1)(\frac{1}{p} - \frac{1}{2})$

$$S^\lambda: L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \rightarrow L^{p/2}(\mathbb{R}^n) \quad \text{is bounded.}$$

Proof. We apply the main Theorem with $(X_{01}, X_{11}) = (X_{02}, X_{12}) := (L_2, L_1)$, $(Y_0, Y_1) := (L^1, L^{1/2})$ and $\mathcal{X}_1 = \mathcal{X}_2$ the space of Schwartz functions on \mathbb{R}^n , which is dense in L^1 and L^2 . We fix $\delta > 0$ and we consider the bilinear analytic family $\{T_z\}_{z \in \bar{S}}$, where $T_z := S^{(n-\frac{1}{2})z+\delta}$ for all $z \in \bar{S}$. We claim that this family is admissible. Indeed, for f, g Schwartz functions we have

$$T_z(f, g)(x) = \iint_{|\xi|^2 + |\eta|^2 \leq 1} (1 - |\xi|^2 - |\eta|^2)^{(n-\frac{1}{2})z+\delta} \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle x, \xi + \eta \rangle} d\xi d\eta,$$

and the map $z \mapsto T_z(f, g)$ is analytic in S , continuous and bounded on \bar{S} , and jointly measurable in (t, x) when $z = it$ or $z = 1 + it$. Moreover, for all $x \in \mathbb{R}^n$ we have

$$\sup_{z \in \bar{S}} \frac{\log |T_z(f, g)(x)|}{e^{\alpha |\operatorname{Im} z|}} < \infty, \quad f, g \in \mathcal{S}$$

with $\alpha = 0 < \pi$; in fact $|T_z(f, g)(x)| \leq \|\widehat{f}\|_{L^1} \|\widehat{g}\|_{L^1}$.

Based on the preceding discussion, we have that when $\operatorname{Re} z = 0$, T_z maps $L^2 \times L^2$ to L^1 with constant $K_0(t) \leq C_{n,\delta} e^{c|t|^2}$ for some $C_{n,\delta}, c > 0$. We also have that when $\operatorname{Re} z = 1$, T_z maps $L^1 \times L^1$ to $L^{1/2}$ with constant $K_1(t) \leq C'_{n,\delta} e^{B|t|^2}$ for some $C'_{n,\delta}, B > 0$. We notice that for these functions $K_i(t)$ we have that the constant $K(\theta, 1, 1/2) < \infty$; in this case $\theta = 2(\frac{1}{p} - \frac{1}{2})$. An application of the main Theorem yields that

$$S^\lambda: L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \rightarrow L^{p/2}(\mathbb{R}^n)$$

provided $\lambda = 2(n - \frac{1}{2})(\frac{1}{p} - \frac{1}{2}) + \delta > (2n - 1)(\frac{1}{p} - \frac{1}{2})$.