Biholomorphic functions on dual of Banach Space

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Notation

- Let $E$ be a complex Banach space.
- $\mathcal{L}(E)$ space of all bounded linear operator
- Let $V$ an open set of $E$, $\mathcal{H}(V, E)$ the space of all holomorphic functions from $V$ into $E$.
- If $f$ belongs to $\mathcal{H}(V, E)$, $df_p$ denotes the derivative of $f$ at the point $p \in V$
Theorem (Carathéodory- Cartan–Kaup-Wu)

If a holomorphic mapping \( f : \Omega \to \Omega \) of a bounded domain \( \Omega \subset \mathbb{C}^n \), satisfies \( f(p) = p \) for some \( p \in \Omega \) and \( |\det\{df_p\}| = 1 \), then \( f \) is a biholomorphic mapping.
Definition

Let $T \in \mathcal{L}(E)$ be a bounded linear operator. $T$ is triangularizable if there is a total, i.e., with dense span, linearly independent sequence $\{e_1, e_2, \ldots, e_n, \ldots\}$ of $E$ such that for all $x \in \text{span} \{e_1, e_2, \ldots, e_n\}$

$$T(x) \in \text{span} \{e_1, e_2, \ldots, e_n\}$$

for every $n \in \mathbb{N}$. In this case, $T(e_k) = \sum_{j=1}^{k} \beta_{j}^{k} e_j$, for all $k = 1, 2, \cdots, n$.

So, the matrix of $T$ when restricted the subspace generated by $\{e_1, e_2, \ldots, e_k\}$ is an upper-triangular matrix, whose main diagonal is given by $\beta_{1}^{1}, \beta_{2}^{2}, \cdots \beta_{k}^{k}$. We shall refer to the sequence $(\beta_{k}^{k})$ as the diagonal entries.
Theorem

Let $H$ be a separable Hilbert space and $\Omega \subset H$, be a bounded convex domain. Fix a point $p \in \Omega$. Let $f : \Omega \rightarrow \Omega$ be a holomorphic mapping such that

- $f(p) = p$;
- the differential $df_p$ is triangularizable;
- $\sigma(df_p) \subset S(0,1)$.

Then $f$ is biholomorphic.

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**Theorem**

Let $E$ be the dual of a Banach space and $G \subset E$ a bounded domain with the separation property $E$ such that its weak* closure coincides with its norm closure. Let $f : G \to G$ be a holomorphic mapping such that

1. $f(p) = p$.
2. $df_p$ is triangularizable with diagonal entries of modulus 1.

Then $f$ is a biholomorphic mapping.
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Definition

An operator is said to be **power bounded** if \( \sup_n \| T^n \| < \infty \).

where \( T^n = T \circ T \circ \cdots \circ T \), is the \( n \)-times self composition of \( T \).

If \( E \) is a Banach space, this is equivalent to \( \sup_n \| T^n(x) \| < \infty \) for each \( x \in E \) according to the Uniform Boundedness Principle.
Theorem

Let \( E \) be a Banach space. Let \( T \in \mathcal{L}(E) \) be a power bounded closed range operator. Assume that:

- There is a subsequence \( (T^{n(k)}) \subset (T^n) \) which pointwise converges to an operator \( S \in \mathcal{L}(E) \).

- If \( S \) is onto and \( T \) one-to-one, or just \( S \) is invertible.

Then \( T \) is an invertible operator and \( \sigma(T) \subset S(0,1) \).

Further, if \( m(k) = n(k+1) - n(k) \), then \( (T^{m(k)}) \) pointwise converges to the identity mapping.

If \( E \) is a Hilbert space, then \( T \) is equivalent to a unitary operator.
The idea of the proof:

Let $y \in E$ arbitrary and consider $x \in E$ such that $Sx = y$. Then

$$y = \lim_n T^n(k)x = \lim T(T^n(k)^{-1}x).$$

Now, for $y_k = T^n(k)^{-1}x$, the sequence $(y_k)$ is a Cauchy sequence since $T$ is an isomorphism. Let $y_0 \in E$ be the limit of $(y_k)$. Then

$$T(y_0) = \lim_n T(y_k) = \lim T(T^n(k)^{-1}(x)) = \lim T^n(k)(x) = y.$$

Therefore $T$ is an onto mapping, thus invertible by the Open Mapping Principle.

In the particular case that $S$ is invertible, also $T$ is injective since if $T(x) = 0$, also $T^n(k)(x) = 0$, so $S(x) = 0$.

It is possible to prove

$$\sup_m \| T^{-m} \| < \infty$$

Indeed, $C := \sup_{n \in \mathbb{Z}} \| T^n \| < \infty$. 
Let \( m(k) = n(k + 1) - n(k) \). It is possible to check that 
\[
\lim_k T^{m(k)} x = x.
\]

Since \( ||T^n|| \leq C \) for all \( n \in \mathbb{Z} \), we have \( r(T) \leq 1 \) and \( r(T^{-1}) \leq 1 \). So \( \sigma(T) \subset \Delta(0,1) \) and \( \sigma(T^{-1}) \subset \Delta(0,1) \). Thus \( \sigma(T) \subset S(0,1) \).
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\[ \lim_k T^{m(k)}x = x. \]

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If $E$ is a Hilbert space, there is an invertible self-adjoint operator $Q$ such that $QTQ^{-1} = U$ is unitary.


**Theorem**

For each power bounded linear operator $T$ in a Hilbert space there is a self-adjoint operator $Q$ such that $Q^{-1}TQ$ is a unitary operator.
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Example

Let \((\lambda_n)_n \subset S(0,1)\). Consider the Hilbert space \(E = \ell_2(\mathbb{R})\). Put \(\mathbb{R} \equiv I \times \mathbb{N}\) for an uncountable set \(I\). For every element in the canonical basis \(\{e_{i,n} : I \times \mathbb{N}\}\) define

\[ T(e_{i,n}) = \lambda_n e_{i,n}. \]

\(T \in \mathcal{L}(E), \text{ and } ||T|| = 1, \text{ and hence } ||T^m|| \leq 1.\)

Since \((\lambda_n)_n \subset S(0,1)\), by the Cantor's diagonalization process there exists a subsequence of positive integer numbers \((m(k))_k\) such that

\[ \lim_k \lambda_n^{m(k)} = 1 \]

for all \(n\). It turns out that \(T^{m_k}\) pointwise converges to \(Id|_E\) since for all pairs \((i, n)\), \(\lim_k T^{m_k}(e_{i,n}) = \lim_k (\lambda_n)^{m_k} e_{i,n} = e_{i,n}.\)

Notice that \(T\) is not triangularizable.
Definition

We say that an open set $A \subset E$ has the separation property if for every $u \in \overline{A} \setminus A$, there is an analytic function $h$ in a neighborhood of $\overline{A}$ such that $h(u) = 1$ and $|h(x)| < 1$ for all $x \in A$.

Examples

- Any convex domain $\Omega$ has the separation property.
- For each $A$ a relatively compact strictly pseudoconvex open set in $\mathbb{C}^n$ with a $C^2$ boundary has the separation property.
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**Examples**

- Any convex domain $\Omega$ has the separation property.
- For each $A$ a relatively compact strictly pseudoconvex open set in $\mathbb{C}^n$ with a $C^2$ boundary has the separation property.
Theorem

Let $E$ be a dual Banach space and let $U \subset E$ a bounded domain with the separation property such that its weak* closure coincides with its norm closure. Let $f \in \mathcal{H}(U, U)$ and $p \in U$ such that

- $f(p) = p$.
- $df_p$ is a one-to-one closed range operator
- there is a subnet $(df_p^{n(k)})$ which converges pointwise to an onto operator.

Then $f$ is a biholomorphic mapping.
Remark

Observe that Theorem does not provide a necessary condition for $f$ to be biholomorphic: Consider $E = \ell_2(\mathbb{Z})$ and $f$ the shift operator in $E$,

$$f((x_n)) = (x_{n+1}).$$

- $f$ is an automorphism of the unit ball,
- $f = df_0$, and $f^m(e_n) = e_{n+m}$,

so the sequence $(f^k(e_n))_k$ does not have a Cauchy subsequence and, therefore, $(f^k)_k$ can not have a pointwise convergent subsequence. In addition, $f$ is known to be not a triangularizable operator.
The idea of the proof:

- Let $G \subset E$ be a bounded domain and $f : G \to G$ be a holomorphic mapping such that there is $p \in G$ with $f(p) = p$. For each $m \in \mathbb{N}$ we have

$$\| (df^m(p)) \| \leq \frac{\sup_{x \in G} \| f(x) \|}{d}$$

where $d = \text{dist}(p, \partial G)$ and $f^m = f \circ f \circ \cdots \circ f$, is the $m$-times self composition of $f$.

There is a constant $C > 0$ such that $\| df^n_p \| < C$ for all $n$. Theorem shows that $df_p$ is invertible, which implies that $df_p^{-1}$ does exist, and also that there is a subsequence $(df_p^{n(k)})$ pointwise convergent to the identity.
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Let $A = \{ f^{n(k)}, k \in I \} \subset \mathcal{H}(U, U)$.

**Lemma**

If $E = X^*$ is a dual Banach space, then for every $A \subset \mathcal{H}(\Omega, E)$ that is bounded for the compact-open topology, there is a compact-open-weak* convergent net to some point $g \in \mathcal{H}(\Omega, E)$.

There is a compact-open-weak* convergent subnet $(f^{n(k_i)})_i$ to some function, say, $g \in \mathcal{H}(U, E)$. Then $g(z) \in \overline{U}^{w^*} = \overline{U}$, for all $z \in U$ and $g(p) = p$. Since $U$ is an open set with separation property we have that $g(U) \subset U$. Moreover, we have that $((df_p)^{n(k_i)})$ is compact-open-weak* convergent to $dg_p$. Therefore $dg_p = I_E$. 
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Theorem

(Cartan) Let $G \subset E$ be an open bounded set, $h \in \mathcal{H}(G,G)$ and $p \in G$, such that $h(p) = p$ and $dh_p = I_E$. Then $h = I_G$.

Using Cartan’s Theorem $g = I_U$.

The bounded subnet $(f^{n(k_i)-1})_i$ has a subnet that converges to a holomorphic function $h \in \mathcal{H}(U,E)$ in the compact-open-weak* topology;

Clearly, $h(p) = p$ and $h(z) \in \overline{U}^{w*} = U$ for all $z \in U$, we have that $h \in \mathcal{H}(U,U)$. Moreover, for $z \in U$,

$$z = \lim f \left( f^{n(k_i)-1} \right) (z) = \lim f^{n(k_i)-1} (f(z)) = h(f(z)),$$

which shows that $h \circ f = I_U$. 

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The bounded subnet $(f^{n(k_i)}-1)_i$ has a subnet that converges to a holomorphic function $h \in \mathcal{H}(U,E)$ in the compact-open-weak* topology.

Clearly, $h(p) = p$ and $h(z) \in \overline{U}^{w^*} = \overline{U}$ for all $z \in U$, we have that $h \in \mathcal{H}(U,U)$. Moreover, for $z \in U$,

$$z = \lim f \left( f^{n(k_i)}-1 \right)(z) = \lim f^{n(k_i)}-1(f(z)) = h(f(z)),$$

which shows that $h \circ f = I_U$. 

Finally, we show that $f \circ h = I_U$.
Since $h \circ f = I_U$, we have

$$dh_p \circ df_p = I_E$$

and since $df_p^{-1}$ exists, it follows that $dh_p = df_p^{-1}$.

Therefore $df_p \circ dh_p = I_E$, using Cartan’s Theorem again, we obtain

$$f \circ h = I_U.$$
Corollary

Let $E$ be a reflexive Banach space and let $U \subset E$ be a convex bounded domain. Let $f \in \mathcal{H}(U, U)$ and $p \in U$ such that

- $f(p) = p$,
- $df_p$ is a one-to-one operator
- There is a subsequence $\left(\left(\left(\left(df_p\right)^{m_k}\right)\right)_k\right)$ that pointwise converges to an onto operator in $\mathcal{L}(E)$.

Then $f$ is biholomorphic.

Since $df_p$ is weakly compact, the result follows.
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Theorem

Let $\Omega \subset \ell_p$ ($p > 1$, $p \neq 2$) be a bounded convex subset and $f : \Omega \rightarrow \Omega$, holomorphic mapping with a fixed point $a \in \Omega$. Suppose that $df_a$ is similar an isometric isomorphism and there is an eigenvector $v = (v_1, v_2, ..., v_i, ..) \in \ell_p$ such that $v_i \neq 0$ for all $i \in \mathbb{N}$, then $f$ is biholomorphic.
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