The Coburn-Simonenko theorem for Toeplitz operators acting between Hardy type subspaces of different Banach function spaces

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NOLIFA, October 17-20, 2017
Classical Hardy spaces

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$ equipped with the normalized Lebesgue measure $dm(t) = |dt|/(2\pi)$. For a complex-valued function $f \in L^1(\mathbb{T})$, let

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})e^{-in\theta} d\theta, \quad n \in \mathbb{Z},$$

be the sequence of its Fourier coefficients.

For $1 \leq p \leq \infty$, consider the Hardy space

$$H^p(\mathbb{T}) := \{ f \in L^p(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0 \}.$$
The Riesz projection

The Riesz projection is the operator $P$ which is defined on $\mathcal{P}$ by

$$P : \sum_{k=-n}^{n} \alpha_k t^k \mapsto \sum_{k=0}^{n} \alpha_k t^k, \quad t \in \mathbb{T}.$$ 

Theorem (Marcel Riesz, 1925)

*If* $1 < p < \infty$, *then* $P$ *extends to a bounded operator*

$$P : L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T}).$$
Toeplitz operators

For $a \in L^\infty(\mathbb{T})$, consider the Toeplitz operator

$$T(a) : H^2(\mathbb{T}) \to H^2(\mathbb{T}) \quad \text{(or) \quad T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T}), \quad 1 < p < \infty)}$$

with symbol $a$ defined by

$$T(a)f = P(af).$$

We are going to extend the study of Toeplitz operators in three directions:

- replace the unit circle by a Jordan rectifiable curve
- replace Lebesgue spaces by arbitrary Banach function spaces
- consider Toeplitz operators between different Hardy type subspaces of Banach function spaces
Jordan rectifiable curves

Let $\Gamma$ be a Jordan curve, that is, a curve that homeomorphic to a circle. We suppose that $\Gamma$ is rectifiable and equip it with the Lebesgue length measure $|d\tau|$ and the counter-clockwise orientation:
Banach function norm I

Let
- $\mathcal{M}(\Gamma)$ be the set of all measurable complex-valued functions on $\Gamma$,
- $\mathcal{M}^+(\Gamma)$ be the subset of functions in $\mathcal{M}(\Gamma)$ whose values lie in $[0, \infty]$,
- $\chi_E$ be the characteristic function of a measurable set $E \subset \Gamma$.

A mapping
$$\rho : \mathcal{M}^+(\Gamma) \to [0, \infty]$$

is called a Banach function norm if,
- for all functions $f, g, f_n \in \mathcal{M}^+(\Gamma)$ with $n \in \mathbb{N}$,
- for all constants $a \geq 0$,
- for all measurable subsets $E$ of $\Gamma$,
the following properties hold:
Banach function norm II (after W. Luxemburg, 1955)

(A1) \( \rho(f) = 0 \iff f = 0 \) a.e.,

\[ \rho(af) = a\rho(f), \]

\[ \rho(f + g) \leq \rho(f) + \rho(g), \]

(A2) \( 0 \leq g \leq f \) a.e. \( \Rightarrow \) \( \rho(g) \leq \rho(f) \) (the lattice property),

(A3) \( 0 \leq f_n \uparrow f \) a.e. \( \Rightarrow \) \( \rho(f_n) \uparrow \rho(f) \) (the Fatou property),

(A4) \( \rho(\chi_E) < \infty \),

(A5) \( \int_E f(\tau) |d\tau| \leq C_E \rho(f) \)

with the constant \( C_E \in (0, \infty) \) that may depend on \( E \) and \( \rho \), but is independent of \( f \).
Banach function spaces

When functions differing only on a set of measure zero are identified, the set $X(\Gamma)$ of all functions $f \in \mathcal{M}(\Gamma)$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\Gamma)$, the norm of $f$ is defined by

$$\|f\|_{X(\Gamma)} := \rho(|f|).$$

The set $X(\Gamma)$ under the natural linear space operations and under this norm becomes a Banach space and

$$L^\infty(\Gamma) \hookrightarrow X(\Gamma) \hookrightarrow L^1(\Gamma).$$
Pointwise multipliers between Banach function spaces

For Banach function spaces $X(\Gamma)$ and $Y(\Gamma)$, let $M(X, Y)$ denote the space of pointwise multipliers from $X(\Gamma)$ to $Y(\Gamma)$ defined by

$$M(X, Y) := \{ f \in M(\Gamma) : fg \in Y(\Gamma) \text{ for all } g \in X(\Gamma) \}.$$ 

It is a Banach space with respect to the operator norm

$$\| f \|_{M(X,Y)} = \sup\{ \| fg \|_{Y(\Gamma)} : g \in X(\Gamma), \| g \|_{X(\Gamma)} \leq 1 \}.$$ 

In particular,

$$M(X, X) \equiv L^\infty(\Gamma).$$
Warining: it may happen that the space $M(X, Y)$ contains only the zero function: if $1 \leq p < q < \infty$, then

$$M(L^p, L^q) = \{0\}.$$ 

If $1 \leq q \leq p \leq \infty$, then $L^p(\Gamma) \hookrightarrow L^q(\Gamma)$ and

$$M(L^p, L^q) \equiv L^r(\Gamma),$$

where $1/r = 1/q - 1/p$. 

Brief (and incomplete) history

Properties of $M(X, Y)$ for general and particular spaces $X(\Gamma)$ and $Y(\Gamma)$ were systematically studied by

- Zabreiko and Rutickii (1976)
- Reisner (1981)
- Maligranda and Persson (1989)
- Nakai (1995)
- Calabuig, Delgado, Sánchez Pérez (2008)
- Maligranda and Nakai (2010)
- Kolwicz, Leśnik, Maligranda (2013-14)
- Leśnik and Tomaszewski (2016)
- Nakai (2016)
Nontrivialty of the space of pointwise multipliers

The continuous embedding

$$L^\infty(\Gamma) \hookrightarrow M(X, Y)$$

holds if and only if

$$X(\Gamma) \hookrightarrow Y(\Gamma).$$
The Cauchy singular integral operator

The Cauchy singular integral of a measurable function \( f : \Gamma \to \mathbb{C} \) is defined by

\[
(Sf)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t, \varepsilon)} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \Gamma,
\]

where the “portion” \( \Gamma(t, \varepsilon) \) is

\[
\Gamma(t, \varepsilon) := \{ \tau \in \Gamma : |\tau - t| < \varepsilon \}, \quad \varepsilon > 0.
\]

It is well known that \((Sf)(t)\) exists a.e. on \( \Gamma \) whenever \( f \) is integrable.
Abstract Hardy spaces built upon Banach function spaces over rectifiable Jordan curves

Lemma (OK, 2003)

If $X(\Gamma)$ is a reflexive Banach function space over a rectifiable Jordan curve $\Gamma$ and the operator $S$ is bounded on $X(\Gamma)$, then the operator

$$P := (I + S)/2$$

is a bounded projection on $X(\Gamma)$, that is, $P^2 = P$.

Consider the Hardy type subspace

$$PX := PX(\Gamma)$$

built upon a Banach function space $X(\Gamma)$.

Note that for $1 < p < \infty$,

$$H^p(\mathbb{T}) = PL^p(\mathbb{T}).$$
Toeplitz operators between different abstract Hardy spaces

The idea to consider $T(a) : H^p(T) \to H^q(T)$ goes back to Vadim Tolokonnikov (1987). It was further developed by Karol Leśnik (2016-2017), who suggested to replace $L^p(T)$ and $L^q(T)$ by rearrangement-invariant spaces $X(T)$ and $Y(T)$.

Let $\Gamma$ be a rectifiable Jordan curve. Assume that $X(\Gamma)$ and $Y(\Gamma)$ are reflexive Banach function spaces and $S$ is bounded on both $X(\Gamma)$ and $Y(\Gamma)$. For $a \in M(X, Y)$, define the Toeplitz operator

$$T(a) : PX \to PY$$

with symbol $a$ by

$$T(a)f = P(af), \quad f \in PX.$$ 

It is clear that $T(a) \in \mathcal{L}(PX, PY)$ and

$$\|T(a)\|_{\mathcal{L}(PX, PY)} \leq \|P\|_{\mathcal{L}(Y)} \|a\|_{M(X, Y)}.$$
Main result

Theorem (OK, November of 2016)

Let $X(\Gamma)$ and $Y(\Gamma)$ be reflexive Banach function spaces over a rectifiable Jordan curve $\Gamma$. Suppose $X \hookrightarrow Y$ and the Cauchy singular integral operator $S$ is bounded on $X$ and on $Y$. If $a \in M(X, Y) \setminus \{0\}$, then

$$T(a) \in \mathcal{L}(PX, PY)$$

has a trivial kernel in $PX$ or a dense image in $PY$.

- For $X(\mathbb{T}) = Y(\mathbb{T}) = L^2(\mathbb{T})$ this result was obtained by Lewis Coburn (1966),
- for $X(\Gamma) = Y(\Gamma) = L^p(\Gamma)$, where $1 < p < \infty$ and $\Gamma$ is a sufficiently smooth Jordan curve, this result was obtained by Igor Simonenko (1968).
Nakano spaces (variable Lebesgue spaces)

Given a rectifiable Jordan curve \( \Gamma \), let \( \mathcal{P}(\Gamma) \) be the set of all measurable functions \( p : \Gamma \rightarrow [1, \infty] \). For \( p \in \mathcal{P}(\Gamma) \) put

\[
\Gamma^{p(\cdot)}_{\infty} := \{ t \in \Gamma : p(t) = \infty \}.
\]

For a measurable function \( f : \Gamma \rightarrow \mathbb{C} \), consider

\[
\varrho_{p(\cdot)}(f) := \int_{\Gamma \setminus \Gamma^{p(\cdot)}_{\infty}} |f(t)|^{p(t)} dt + \|f\|_{L^\infty(\Gamma^{p(\cdot)}_{\infty})}.
\]

The Nakano space (= variable Lebesgue space) \( L^{p(\cdot)}(\Gamma) \) is defined as the set of all measurable functions \( f : \Gamma \rightarrow \mathbb{C} \) such that \( \varrho_{p(\cdot)}(f/\lambda) < \infty \) for some \( \lambda > 0 \). This space is a Banach function space with respect to the Luxemburg-Nakano norm given by

\[
\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf \{ \lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1 \}.
\]

If \( p \in \mathcal{P}(\Gamma) \) is constant, then \( L^{p(\cdot)}(\Gamma) \) is nothing but the standard Lebesgue space \( L^p(\Gamma) \).
Pointwise multipliers between Nakano spaces

**Theorem (Nakai, 2016 (under some additional condition) and OK, January of 2017)**

Let $\Gamma$ be a rectifiable Jordan curve. Suppose $p, q, r \in \mathcal{P}(\Gamma)$ are related by

$$\frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}, \quad t \in \Gamma.$$ 

Then

$$M(L^{p(\cdot)}, L^{q(\cdot)}) = L^{r(\cdot)}(\Gamma).$$

Nakai additionally assumed that

$$\sup_{t \in \Gamma \setminus \Gamma^r} r(t) < \infty.$$
Let \( \Gamma \) be a rectifiable Jordan curve. We say that an exponent \( p \in \mathcal{P}(\Gamma) \) is locally log-Hölder continuous if

\[
1 < p_- := \operatorname{ess inf} \limits_{t \in \Gamma} p(t) \leq p_+ := \operatorname{ess sup} \limits_{t \in \Gamma} p(t) < \infty
\]

and there exists a constant \( C_{p(\cdot),\Gamma} \in (0, \infty) \) such that

\[
|p(t) - p(\tau)| \leq \frac{C_{p(\cdot),\Gamma}}{-\log |t - \tau|} \quad \text{for all} \quad t, \tau \in \Gamma \text{ satisfying } |t - \tau| < 1/2.
\]

The class of all locally log-Hölder continuous exponent will be denoted by \( LH(\Gamma) \).

**Theorem (Kokilashvili, Paatashvili, Samko, 20006)**

Let \( \Gamma \) be a rectifiable Jordan curve and \( p \in LH(\Gamma) \). Then the Cauchy singular integral operator \( S \) is bounded on \( L^{p(\cdot)} \) if and only if \( \Gamma \) is a Carleson curve, that is,

\[
\sup \limits_{t \in \Gamma} \sup \limits_{\varepsilon > 0} \frac{|\Gamma(t, \varepsilon)|}{\varepsilon} < \infty.
\]
The Coburn-Simonenko theorem for Nakano spaces

Theorem (OK, January of 2017)

Let $\Gamma$ be a Carleson Jordan curve. Suppose variable exponents $p, q \in LH(\Gamma)$ and $r \in \mathcal{P}(\Gamma)$ are related by

$$\frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}, \quad t \in \Gamma.$$ 

If $a \in L^{r(\cdot)}(\Gamma) \setminus \{0\}$, then the Toeplitz operator $T(a) \in \mathcal{L}(PL^{p(\cdot)}, PL^{q(\cdot)})$ has a trivial kernel in $PL^{p(\cdot)}$ or a dense image in $PL^{q(\cdot)}$. 

Corollary (OK, November of 2016)

Let $1 < q \leq p < \infty$ and $1/r = 1/q - 1/p$. If $a \in L^r(\mathbb{T}) \setminus \{0\}$, then the Toeplitz operator

$$T(a) \in \mathcal{L}(H^p(\mathbb{T}), H^q(\mathbb{T}))$$

has a trivial kernel in $H^p(\mathbb{T})$ or a dense image in $H^q(\mathbb{T})$.

I have never seen this result in the literature explicitly stated. May I suppose that it is new?