Generalized Lorentz spaces and Köthe duality

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μ measure on the the measure space $(\Omega, \mathcal{A}, \mu)$
$L^0(\Omega) = L^0(\Omega, \mathcal{A}, \mu)$, μ-measurable real valued functions on Ω
$L^0_+(\Omega)$ non-negative functions from $L^0(\Omega)$.
$L_1 = L_1(\Omega)$, $\|f\|_1$, $L_\infty = L_\infty(\Omega)$, $\|f\|_\infty$
$L_1 + L_\infty(\Omega)$, $\|f\|_{L_1+L_\infty(\Omega)} = \inf\{\|g\|_1 + \|h\|_\infty : f = g + h\} < \infty$
$L_1 \cap L_\infty(\Omega)$, $\|f\|_{L_1 \cap L_\infty(\Omega)} = \max\{\|f\|_1, \|f\|_\infty\} < \infty$.

A Banach function space E over $(\Omega, \mathcal{A})$, is a complete vector space $E \subset L^0(\Omega)$ equipped with a norm $\|\cdot\|_E$ such that if $0 \le f \le g$, where $g \in E$ and $f \in L^0(\Omega)$, then $f \in E$ and $\|f\|_E \le \|g\|_E$.

The space E satisfies the **Fatou property** whenever for any $f \in L^0(\Omega)$, $f_n \in E$ such that $f_n \uparrow f$ a.e. and $\sup \|f_n\|_E < \infty$ it follows that $f \in E$ and $\|f_n\|_E \uparrow \|f\|_E$.

distribution of f with respect to μ, $d_f^\mu(s) = \mu\{|f| > s\}$, $s \ge 0$, and its **decreasing rearrangement** $f^{*,\mu}(t) = \inf\{s > 0 : d_g^\mu(s) \le t\}$, $0 < t < \mu(\Omega)$.
f, g are **equimeasurable** (with respect to the measures μ and ν) if $d_f^\mu(s) = d_g^\nu(s)$, $s \ge 0$; equivalently, $f^{*,\mu} = g^{*,\nu}$.

A Banach function space E is **symmetric** space (with respect to μ) whenever $\|f\|_E = \|g\|_E$ for every μ-equimeasurable functions $f, g \in E$.

fundamental function of a symmetric space E is $\phi_E(t) = \|\chi_A\|_E$, $\mu(A) = t$.

support of the symmetric space E is the entire set Ω whenever $\chi_A \in E$ for any $A \in \mathcal{A}$ with $\mu(A) < \infty$.

Hardy-Littlewood order $f \prec g$ for $f, g \in L_1 + L_\infty(\Omega)$, $\int_0^x f^* \leq \int_0^x g^*$ for every $x \in (0, \mu(\Omega))$; $(f+g) \prec f^* + g^*$.

E **fully symmetric** if E is symmetric and if for any $f \in L^0(\Omega)$ and $g \in E$ with $f \prec g$ we have that $f \in E$ and $\|f\|_E \leq \|g\|_E$.

Köthe dual space E' of E is the set of $f \in L^0(\Omega)$ such that

$$\|f\|_{E'} = \sup\left\{\int_\Omega |fg|\, d\mu : \|g\|_E \leq 1\right\} < \infty.$$

$(E', \|\cdot\|_{E'})$ is Banach function space with the Fatou property. If E is symmetric then E' is fully symmetric, and

$$\|f\|_{E'} = \sup\left\{\int_0^{\mu(\Omega)} f^* g^*\, dm : \|g\|_E \leq 1\right\}.$$

$I = (0, a)$, $0 < a \leq \infty$, $L^0 = L^0(I)$
If $\Omega = I$ and $\mu = m$ then $d_f^m = d_f$, $f^* = f^{*,m}$.

weight function $w : I \to (0, \infty)$ decreasing positive
measure on I**,** $\omega = wdm$, $\omega(A) = \int_A w$ for $A \subset I$
d_f^w and $f^{*,w}$, **distribution** and **decreasing rearrangement** of f w. r. to the measure $\omega = wdm$

$$W(t) = \int_0^t w\,dm, \quad t \in I, \quad W(\infty) = \int_0^\infty w\,dm \quad \text{if} \quad I = (0, \infty)$$

w is **regular** if $W(t) \leq Ctw(t)$ for some $C \geq 1$ and all $t \in I$

Let $b = \omega(I) = W(a) \in (0, +\infty]$ and $J = (0, b)$
If $a = \infty$ then $I = (0, \infty) = J$
J is equipped with the Lebesgue measure m

symbol E will always stand for a **fully symmetric Banach function space** $E \subset L^0(J)$, **support** of E is equal to J

weighted E space, $E_w \subset L^0 = L^0(I)$,

$$E_w = \{f \in L^0 : f^{*,w} \in E\}, \quad \|f\|_{E_w} = \|f^{*,w}\|_E, \;\; f \in E_w.$$

If $f \in L^0(I)$ then $f^{*,w} \in L^0(J)$. $1 \le p < \infty$, then $E_w = (L_p)_w$ is a **weighted L_p space** on (I, ω). $f \in E_w = (L_p)_w$,

$$\|f\|_{(L_p)_w} = (\int_J (f^{*,w})^p dm)^{1/p} = (\int_J (|f|^p)^{*,w} dm)^{1/p}$$
$$= (\int_I |f|^p d\omega)^{1/p} = (\int_I |f|^p w dm)^{1/p}.$$

Analogously for the Orlicz space $E = L_\varphi(J)$, $E_w = (L_\varphi)_w$ is a **weighted Orlicz space** associated with the Orlicz modular

$$\int_J \varphi(f^{*,w})\, dm = \int_I \varphi(|f|)\, w dm.$$

E_w is a Banach function space in $L^0(I)$, **not symmetric w. r. to m, isometrically order isomorphic to E on** (J, m).

Lemma

Assume that $W < \infty$ *on* I. *Then*
(i) *Every* $f \in L^0$ *is equimeasurable with respect to* ω, *to* $f \circ W^{-1}$ *with respect to* m. *Consequently,*

$$(f \circ W^{-1})^* = f^{*,w}.$$

(ii) $f \in L^0$ *belongs to* E_w *if and only if* $f \circ W^{-1}$ *belongs to* E, *and then*

$$\|f\|_{E_w} = \|f \circ W^{-1}\|_E.$$

Consequently, E_w *is an order ideal in* L^0 *and the map* $f \mapsto f \circ W^{-1}$ *induces an order isometry from* E_w *onto* E.

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generalized Lorentz space $\Lambda_{E,w}$, the **symmetrization** of E_w,

$$\Lambda_{E,w} = \{f \in L^0 : f^* \in E_w\}, \quad \|f\|_{\Lambda_{E,w}} = \|f^*\|_{E_w}.$$

If $W(t) = \infty$ for $t > 0$, then $\Lambda_{E,w} = \{0\}$ or $\Lambda_{E,w} = L_\infty(I)$.

Assume that $W < \infty$ on I.

If $E = L_\varphi(J)$ is **Orlicz space** then $\Lambda_{E,w} = \Lambda_{\varphi,w}$ is the **Orlicz-Lorentz space**; $\|f\|_{\Lambda_{\varphi,w}} = \|f^*\|_{L_{\varphi,w}}$.

If $\varphi(t) = t^p$, $1 \le p < \infty$, $E = L_p(J)$ then $\Lambda_{E,w} = \Lambda_{p,w}$ is the **Lorentz space**; $\|f\|_{\Lambda_{p,w}} = (\int_I f^{*p} w)^{1/p}$.

If $E = L_\infty(J)$ then $E_w = L_\infty(I) = \Lambda_{E,w}$.

Theorem

Let $W < \infty$ on I.

(i) *The support of $\Lambda_{E,w}$ is I.*

(ii) *For all $f \in \Lambda_{E,w}$,*

$$\|f\|_{\Lambda_{E,w}} = \|f^* \circ W^{-1}\|_E.$$

(iii) *The functional $\|\cdot\|_{\Lambda_{E,w}}$ is a norm, and the Lorentz space $\Lambda_{E,w}$ is a fully symmetric Banach space. If E has the Fatou property then $\Lambda_{E,w}$ has also this property.*

Proof.

(iii) $D_2W^{-1} \leq (1/2)W^{-1}$, $D_2f(t) = f(t/2)$. $f \in \Lambda_{E,w} \Rightarrow f^* \circ W^{-1} \in E$, and $D_2 f \in \Lambda_{E,w}$ since D_2 is a bounded operator in E, and

$$(D_2f)^* \circ W^{-1} = (D_2f^*) \circ W^{-1} = f^* \circ ((1/2)W^{-1}) \\ \leq f^* \circ (D_2W^{-1}) = D_2(f^* \circ W^{-1}) \in E.$$

$f, g \in \Lambda_{E,w} \Rightarrow (f+g)^* \circ W^{-1} \leq (D_2(f+g))^* \circ W^{-1} \leq D_2(f^* \circ W^{-1}) + D_2(g^* \circ W^{-1}) \in E \Rightarrow f+g \in \Lambda_{E,w}$.

Let $f \in L^0$, , $g \in \Lambda_{E,w}$ with $f \prec g$, and $x \in J$.Then

$$\int_0^x f^* \circ W^{-1} = \int_0^{W^{-1}(x)} f^* w \leq \int_0^{W^{-1}(x)} g^* w = \int_0^x g^* \circ W^{-1}.$$

Hence $f^* \circ W^{-1} \prec g^* \circ W^{-1}$. By full symmetry of E, $f^* \circ W^{-1} \in E$,

$$\|f\|_{\Lambda_{E,w}} = \|f^* \circ W^{-1}\|_E \leq \|g^* \circ W^{-1}\|_E = \|g\|_{\Lambda_{E,w}},$$

so $\|\cdot\|_{\Lambda_{E,w}}$ is fully symmetric. □

Proof.

We also have $\|f+g\|_{\Lambda_{E,w}} \le \|f\|_{\Lambda_{E,w}} + \|g\|_{\Lambda_{E,w}}$ by $(f+g)^* \prec f^* + g^*$.

The normed function space $\Lambda_{E,w}$ is **complete** since it is a **symmetrization** of the complete space E_w.

Suppose now that E has the **Fatou property**. Take $f_n, f \in L^0(J)$, $f_n \uparrow f$ a.e., and $\sup \|f_n\|_{\Lambda_{E,w}} < \infty$. Then $f_n^* \circ W^{-1} \uparrow f^* \circ W^{-1}$ a.e., and by (ii) $\sup \|f_n^* \circ W^{-1}\|_E = \sup \|f_n\|_{\Lambda_{E,w}} < \infty$. Now by the Fatou property of E, $f^* \circ W^{-1} \in E$ and
$\|f_n\|_{\Lambda_{E,w}} = \|f_n^* \circ W^{-1}\|_E \uparrow \|f^* \circ W^{-1}\|_E = \|f\|_{\Lambda_{E,w}}$. □

Classes $M_{E,w}$

Define a class $M_{E,w}$ contained in $L^0 = L^0(I)$ which will be used for finding the Köthe dual of the Lorentz space $\Lambda_{E,w}$.

Let the **class** $M_{E,w}$ and the **gauge** on $M_{E,w}$ be defined by

$$M_{E,w} = \left\{ f \in L^0 : \frac{f^*}{w} \in E_w \right\} \quad \text{and} \quad \|f\|_{M_{E,w}} = \left\| \frac{f^*}{w} \right\|_{E_w} = \left\| \left(\frac{f^*}{w} \right)^{*,w} \right\|_E .$$

Theorem

(i) *The class* $M_{E,w}$ *is a solid symmetric subset of* L^0*, that is* $\|f\|_{M_{E,w}} = \|f^*\|_{M_{E,w}}$ *and if* $f \in L^0$, $g \in M_{E,w}$ *and* $|f| \leq |g|$ *a.e. then* $f \in M_{E,w}$ *and* $\|f\|_{M_{E,w}} \leq \|g\|_{M_{E,w}}$.

(ii) *For all* $x \in I$, $\chi_{(0,x)} \in M_{E,w}$. *Consequently the support of* $M_{E,w}$ *is equal to the entire interval* I.

(iii) *The fundamental function* $\phi_{M_{E,w}}(x) = \|\chi_{(0,x)}\|_{M_{E,w}}$, $x \in I$, *verifies*

$$\phi_{M_{E,w}}(x) \leq 2\phi_E(1 \wedge b)\left(x + \frac{1}{w(x)}\right).$$

(iv) *If* $W < \infty$ *on* I*, then*

$$f \in M_{E,w} \iff \frac{f^*}{w} \circ W^{-1} \in E \quad \text{and} \quad \|f\|_{M_{E,w}} = \left\|\frac{f^*}{w} \circ W^{-1}\right\|_E.$$

(v) *If* E *has the Fatou property then the class* $M_{E,w}$ *has this property, that is for every* $f \in L^0$, $0 \leq f_n \in M_{E,w}$ *with* $f_n \uparrow f$ *a.e. and* $\sup_n \|f_n\|_{M_{E,w}} = K < \infty$ *we have* $f \in M_{E,w}$ *and* $\|f\|_{M_{E,w}} = K$.

Theorem

For any $f \in M_{E,w}$ *we have*

$$\left\| \frac{f^*}{w} \right\|_{E_w} = \|f\|_{M_{E,w}} = \inf \left\{ \left\| \frac{f}{v} \right\|_{E_v} : v \geq 0, v^* = w, \operatorname{supp} v \supset \operatorname{supp} f \right\}$$

with the convention that $\|g\|_E = \infty$ *for every* $g \notin E$*, and* $f(t)/v(t) = 0$ *whenever* $f(t) = 0$*.*

Moreover if $W < \infty$ *on* I*, then for* $f \in L^0$ *we have that* $f \in M_{E,w}$ *if and only if* $\frac{f}{v} \circ V^{-1} \in E$ *for some* $v \geq 0$ *with* $v^* = w$ *and* $\operatorname{supp} v \supset \operatorname{supp} f$*.*

Theorem

Let $v \in L^0_+$ *be such that* $v^* = w$. *Assume* $f \in L_1 + L_\infty(I)$ *with* $\operatorname{supp} f \subset \operatorname{supp} v$. *Then*

$$\left(\frac{f^*}{w}\right)^{*,w} \prec \left(\frac{f}{v}\right)^{*,v}.$$

In particular if $f/v \in E_v$ *then* $f^*/w \in E_w$ *and* $\|f^*/w\|_{E_w} \leq \|f/v\|_{E_v}$.

We prove first two lemmas.

Lemma

A. *For any* $f, g \in L^0_+$ *we have* $(f \wedge g)^* \le f^* \wedge g^*$.

Lemma

B. *For every* $f, g \in L^0_+$ *such that* $f^*, g^* < \infty$, *it holds*

$$\int_I (f^* - g^*)_+ dm \le \int_I (f - g)_+ dm.$$

Using Lemma **A** and Lorentz-Shimogaki inequality for rearrangements, we obtain in fact the more powerful result

$$(f^* - g^*)_+ \prec (f - g)_+.$$

Indeed since $f \ge f \wedge g$, Lorentz-Shimogaki's theorem gives $f^* - (f \wedge g)^* \prec f - f \wedge g$ and

$$(f^* - g^*)_+ = f^* - f^* \wedge g^* \le f^* - (f \wedge g)^* \prec f - f \wedge g = (f - g)_+.$$

However Lemma **B**, which requires only quite elementary ingredients in its proof, will suffice for our purpose.

$f \in L_1 + L_\infty(\Omega)$, $x \in (0, \mu(\Omega))$,

$$\int_0^x f^{*,\mu}\, dm = \inf\{\|g\|_1 + x\|h\|_\infty : g \in L_1, h \in L_\infty, f = g + h\}$$

$$= \inf_{\lambda > 0} \left[\int (|f| - \lambda)_+ \, d\mu + \lambda x \right]$$

Proof of Theorem.

By Lemma **B**, for every $\lambda > 0$ we have

$$\int_I \left(\frac{f^*}{w} - \lambda\right)_+ w\,dm = \int_I (f^* - \lambda w)_+\,dm = \int_I (f^* - (\lambda v)^*)_+\,dm$$
$$\leq \int_I (f - \lambda v)_+\,dm = \int_I \left(\frac{f}{v} - \lambda\right)_+ v\,dm.$$

For any $x \in J$,

$$\int_0^x \left(\frac{f^*}{w}\right)^{*,w} dm = \inf_{\lambda>0}\left[\int_I \left(\frac{f^*}{w} - \lambda\right)_+ wdm + \lambda x\right]$$
$$\leq \inf_{\lambda>0}\left[\int_I \left(\frac{f^*}{v} - \lambda\right)_+ v\,dm + \lambda x\right] = \int_0^x \left(\frac{f^*}{v}\right)^{*,v} dm,$$

and the proof is completed. □

Theorem

Let $f \in L^0$ *have a finite decreasing rearrangement* f^*. *If* I *is a finite interval* $(0, a)$, *or* $I = (0, \infty)$ *and* $\lim_{t \to \infty} f^*(t) = 0$, *then there exists* $v \in L^0_+$ *such that*

$$v^* = w, \quad \operatorname{supp} v \supset \operatorname{supp} f \;\; \textit{and} \;\; \left(\frac{f^*}{w}\right)^{*,w} = \left(\frac{f}{v}\right)^{*,v}.$$

If $I = (0, \infty)$ *and* $\lim_{t \to \infty} f^*(t) > 0$ *then for every* $\epsilon > 0$ *there exists* $0 < v \in L^0$ *such that*

$$v^* = w \quad \textit{and} \quad \left(\frac{f}{v}\right)^{*,v} \leq (1 + \epsilon) \left(\frac{f^*}{w}\right)^{*,w}.$$

Proof.

Let first $I = (0, a)$, $a < \infty$, then there exists an onto and measure preserving transformation $\tau : I \to I$ such that $|f|(t) = f^* \circ \tau(t)$, $t \in I$. Setting $v = w \circ \tau$, we have $v > 0$, $v^* = w$, and for every scalar $\lambda > 0$,

$$\nu\left\{\frac{|f|}{v} > \lambda\right\} = \int_I \chi_{(\lambda,\infty)} \circ \left(\frac{|f|}{v}\right) v\, dm = \int_I \chi_{(\lambda,\infty)} \circ \left(\frac{f^* \circ \tau}{w \circ \tau}\right) w \circ \tau\, dm$$

$$= \int_I \chi_{(\lambda,\infty)} \circ \left(\frac{f^*}{w}\right) wdm = \omega\left\{\frac{f^*}{w} > \lambda\right\},$$

and the desired equality of rearrangements follows. □

Proof.

$I = (0,\infty)$ and $\lim_{t\to\infty} f^*(t) = 0$. There exists a set $A_f \supset \operatorname{supp} f$ (in fact $A_f = \operatorname{supp} f$ if $m(\operatorname{supp} f) = \infty$, and $A_f = (0,\infty)$ if $m(\operatorname{supp} f) < \infty$), and a measure preserving transformation $\tau : A_f \to (0,\infty)$ which is onto such that $|f(t)| = f^* \circ \tau(t)$ for $t \in A_f$. Define $v(t) = w \circ \tau(t)$ for $t \in A_f$, and $v(t) = 0$ otherwise. Then we have $\operatorname{supp} v = A_f \supset \operatorname{supp} f$ and again $v^* = w$. Moreover for every scalar $\lambda > 0$,

$$\nu\left\{\frac{|f|}{v} > \lambda\right\} = \int_{A_f} \chi_{(\lambda,\infty)} \circ \left(\frac{|f|}{v}\right) v\, dm = \int_{A_f} \chi_{(\lambda,\infty)} \circ \left(\frac{f^* \circ \tau}{w \circ \tau}\right) w \circ \tau\, dm$$

$$= \int_I \chi_{(\lambda,\infty)} \circ \left(\frac{f^*}{w}\right) w dm = \omega\left\{\frac{f^*}{w} > \lambda\right\},$$

which shows the equality in the hypothesis.

□

If $W < \infty$ and $0 \le v \in L^0(I)$ with $v^* = w$, then the above results may be restated in a more transparent way.

Corollary

If $W < \infty$ on I, then for any $v \in L^0_+$ with $v^* = w$, and every $f \in L_1 + L_\infty(I)$ with $\operatorname{supp} f \subset \operatorname{supp} v$ we have

$$\frac{f^*}{w} \circ W^{-1} \prec \frac{f}{v} \circ V^{-1}.$$

Moreover if $I = (0, a)$ with $a < \infty$ or if $I = (0, \infty)$ and $\lim_{t\to\infty} f^*(t) = 0$, then there exists $v \in L^0_+$ with $\operatorname{supp} f \subset \operatorname{supp} v$ such that $v^* = w$ and

$$\frac{f^*}{w} \circ W^{-1} = \frac{f}{v} \circ V^{-1}.$$

If $I = (0, \infty)$ and $\lim\limits_{t\to\infty} f^*(t) > 0$ then for every $\epsilon > 0$ there exists $v > 0$ on I such that $v^* = w$ and

$$\frac{f^*}{w} \circ W^{-1} \le \frac{f}{v} \circ V^{-1} \le (1+\epsilon)\,\frac{f^*}{w} \circ W^{-1}.$$

The class $M_{E,w}$ does not need to be either linear or normable. Before we prove the **main result on normability** of the class $M_{E,w}$ we need the following lemma.

Lemma

Let w_1, w_2 be two decreasing weights on I such that for some constant $C \geq 1$ it holds that $w_1 \leq Cw_2$ a.e.. Then for every function f we have $\left(\frac{f}{w_2}\right)^{*,w_2} \prec C\left(\frac{f}{w_1}\right)^{*,w_1}$.
Consequently if $\int_I w_1 dm = \int_I w_2 dm = b$ and E is a fully symmetric space on $J = (0, b)$ then $M_{E,w_1} \subset M_{E,w_2}$ and moreover $\|f\|_{M_{E,w_2}} \leq C\|f\|_{M_{E,w_1}}$ whenever $f \in M_{E,w_1}$.

Theorem

Assume that the **weight** w **is regular** *that is* $W(t) \leq Ctw(t)$ *for some* $C \geq 1$ *and all* $t \in I$. *Then* $M_{E,w}$ *is a vector space and there exists a lattice norm* $|||\cdot|||$ *on* $M_{E,w}$ *such that*

$$|||f||| := \inf\left\{\sum_{i=1}^{n} \|f_i\|_{M_{E,w}} : \sum_{i=1}^{n} |f_i| \geq |f|\right\} \leq \|f\|_{M_{E,w}} \leq C|||f|||. \qquad (1)$$

Consequently the class $M_{E,w}$ *is a normable vector lattice.*

Proof.

We will prove that for any finite family $f_1, \ldots, f_n$ in $M_{E,w}$ we have

$$\left\| \sum_{i=1}^{n} f_i \right\|_{M_{E,w}} \leq C \sum_{i=1}^{n} \|f_i\|_{M_{E,w}}, \tag{2}$$

where C is the constant of regularity of w. Then $||| \cdot |||$ defined by (1) is a vector lattice norm on $M_{E,w}$ equivalent to the gauge $\|f\|_{M_{E,w}}$.
We claim that

$$\left(\frac{1}{w} \left(\sum_{i=1}^{n} f_i \right)^* \right) \circ W^{-1} \prec C \sum_{i=1}^{n} \left(\frac{f_i}{v_i} \circ V_i^{-1} \right)^* \tag{3}$$

for every non-negative functions $v_1, \ldots, v_n$ with $\operatorname{supp} f_i \subset \operatorname{supp} v_i$, $v_i^* = w$, $i = 1, \ldots, n$, where
$V_i(t) = \int_0^t v_i \, dm \leq \int_0^t v_i^* \, dm = \int_0^t w \, dm = W(t) < \infty$ for all $t \in I$. □

The statement of the claim then implies the following

$$\left\| \left(\frac{1}{w} \left(\sum_{i=1}^{n} f_i \right)^* \right) \circ W^{-1} \right\|_E \leq C \sum_{i=1}^{n} \left\| \frac{f_i}{v_i} \circ V_i^{-1} \right\|_E .$$

Taking the infimum of every right term with respect to v_i with $v_i^* = w$ and $\operatorname{supp} f_i \subset \operatorname{supp} v_i$ for $i = 1, \ldots, n$, we get by Proposition 4,

$$\left\| \left(\frac{1}{w} \left(\sum_{i=1}^{n} f_i \right)^* \right) \circ W^{-1} \right\|_E \leq C \sum_{i=1}^{n} \left\| \frac{f_i^*}{w} \circ W^{-1} \right\|_E ,$$

and consequently in view of Proposition 3(iv) we obtain the desired inequality (2).

Now in order to finish it is enough to prove claim (3), which is equivalent to the following inequality

$$\int_0^x \left(\frac{\left(\sum_{i=1}^n f_i\right)^*}{w} \circ W^{-1} \right)^* dm \leq C \sum_{i=1}^n \int_0^x \left(\frac{|f_i|}{v_i} \circ V_i^{-1} \right)^* dm, \quad x \in J. \tag{4}$$

For any measurable $v \geq 0$ with $V(t) = \int_0^t v\, dm < \infty$, $t \in I$, and $f \in L^0$ such that $\operatorname{supp} f \subset \operatorname{supp} v$, by equimeasurability of f/v for $\nu = vdm$ and $(f/v) \circ V^{-1}$ for m we have that $(f/v)^{*,v} = ((f/v) \circ V^{-1})^*$.
Hence by (5) for any $x \in J$,

$$\int_0^x \left(\frac{f}{v} \circ V^{-1} \right)^* dm = \int_0^x \left(\frac{f}{v} \right)^{*,v} dm = \inf_{\lambda > 0} \left\{ \left\| \left(\frac{|f|}{v} - \lambda \right)_+ \right\|_{L_1(v)} + \lambda x \right\}$$

$$= \inf_{\lambda > 0} \left\{ \int_I (|f| - \lambda v)_+ dm + \lambda x \right\}.$$

Thus the righthand side of (4) has the following form

$$R(x) := \sum_{i=1}^{n} \int_0^x \left(\frac{|f_i|}{v_i} \circ V_i^{-1} \right)^* dm$$
$$= \inf_{\lambda_1,\ldots,\lambda_n>0} \left\{ \int_I \sum_{i=1}^{n} (|f_i| - \lambda_i v_i)_+ dm + \sum_{i=1}^{n} \lambda_i x \right\}.$$

The function $(s,t) \mapsto (|s| - t)_+$ is positively homogeneous and convex on $\mathbb{R}_2$. Hence a.e. on I,

$$\left(\left| \sum_{i=1}^{n} f_i \right| - \sum_{i=1}^{n} \lambda_i v_i \right)_+ \leq \left(\sum_{i=1}^{n} |f_i| - \sum_{i=1}^{n} \lambda_i v_i \right)_+$$
$$= n \left(\sum_{i=1}^{n} \frac{1}{n} (|f_i| - \lambda_i v_i) \right)_+ \leq \sum_{i=1}^{n} (|f_i| - \lambda_i v_i)_+.$$

Thus by (6), in view of (5) we get for $x \in J$,

$$R(x) \geq \inf_{\lambda_1,\dots,\lambda_n>0} \left[\int_I \left(\left| \sum_{i=1}^n f_i \right| - \sum_{i=1}^n \lambda_i v_i \right)_+ dm + x \sum_{i=1}^n \lambda_i \right]$$

$$= \inf_{\substack{\alpha_1,\dots,\alpha_n>0 \\ \sum \alpha_i=1}} \inf_{\lambda>0} \left[\int_I \left(\left| \sum_{i=1}^n f_i \right| - \lambda \sum_{i=1}^n \alpha_i v_i \right)_+ dm + \lambda x \right]$$

$$= \inf_{\substack{\alpha_1,\dots,\alpha_n>0 \\ \sum \alpha_i=1; v=\sum \alpha_i v_i}} \inf_{\lambda>0} \left[\int_I \left(\frac{\left| \sum_{i=1}^n f_i \right|}{v} - \lambda \right)_+ v\, dm + \lambda x \right]$$

$$= \inf_{v\in \mathrm{conv}(v_1,\dots,v_n)} \int_0^x \left(\frac{\sum_{i=1}^n f_i}{v} \right)^{*,v} dm$$

$$= \inf_{v\in \mathrm{conv}(v_1,\dots,v_n)} \int_0^x \left(\frac{\left| \sum_{i=1}^n f_i \right|}{v} \circ V^{-1} \right)^* dm.$$

If $v \in \mathrm{conv}(v_1, \ldots, v_n)$ we have $v = \sum_{i=1}^n \alpha_i v_i$ for some $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$. Since by $v_i^* = w$ we have $V_i(t) \leq W(t)$ for every $0 \leq t < a$, with equality at the limit $t \to a$, we obtain $V(t) = \sum_{i=1}^n \alpha_i V_i(t) \leq \sum_{i=1}^n \alpha_i W(t)$ for $t \in I$ with equality at the limit $t \to a$, so that the continuous function V maps I onto J, and we may define V^{-1} as in the proof of Corollary 9. We have also $v^* \prec \sum_{i=1}^n \alpha_i v_i^* = w$, hence

$$tv^*(t) \leq \int_0^t v^* \leq W(t) \leq Ctw(t), \quad t \in I,$$

by regularity of w.

But then for every $v \in \mathrm{conv}(v_1, \ldots, v_n)$, letting $V_*(t) = \int_0^t v^*$, we get for $x \in J$,

$$\int_0^x \left(\frac{\left| \sum_{i=1}^n f_i \right|}{v} \circ V^{-1} \right)^* dm \geq \int_0^x \left(\frac{\left(\sum_{i=1}^n f_i \right)^*}{v^*} \circ V_*^{-1} \right)^* dm$$
$$\geq \frac{1}{C} \int_0^x \left(\frac{\left(\sum_{i=1}^n f_i \right)^*}{w} \circ W^{-1} \right)^* dm =: L(x),$$

where the first inequality results from Corollary 9 with v^*, V_* playing the role of w, W respectively, and the second one by Lemma 10 applied to the weights v^* and w. Thus $CR(x) \succ L(x)$, and this proves the claim and completes the proof.

Köthe duality of $M_{E,w}$

The Köthe dual of the class $M_{E,w}$ is defined analogously like in the case of a Banach function spaces, as the set of elements $f \in L^0$ such that

$$\|f\|_{(M_{E,w})'} := \sup \left\{ \int_I |fg| \, dm : g \in M_{E,w}, \|g\|_{M_{E,w}} \leq 1 \right\} < \infty,$$

This set is an order ideal in L^0 on which $f \mapsto \|f\|_{(M_{E,w})'}$ defines a vector lattice norm. Equipped with this norm, the space $(M_{E,w})'$ becomes a Banach function space.

Theorem

If $W < \infty$ on I, then the Köthe dual $(M_{E,w})' = \Lambda_{E',w}$ isometrically, that is $\|f\|_{(M_{E,w})'} = \|f\|_{\Lambda_{E',w}}$.

Proof.

The proof will be done in several steps.

a) $\Lambda_{E',w} \subset (M_{E,w})'$, $\|f\|_{(M_{E,w})'} \leq \|f\|_{\Lambda_{E',w}}$.

If $f \in \Lambda_{E',w}, g \in M_{E,w}$ then

$$\int_I |fg|\, dm \leq \int_I f^* g^*\, dm = \int_I f^* \frac{g^*}{w} w dm = \int_J (f^* \circ W^{-1}) \left(\frac{g^*}{w} \circ W^{-1} \right) dm$$
$$\leq \|f^* \circ W^{-1}\|_{E'} \left\| \frac{g^*}{w} \circ W^{-1} \right\|_E = \|f^*\|_{(E')_w} \left\| \frac{g^*}{w} \right\|_{E_w}$$
$$= \|f\|_{\Lambda_{E',w}} \|g\|_{M_{E,w}}.$$

□

Corollary

Let $W < \infty$ on I. If E has the Fatou property and w is regular, then

$$(\Lambda_{E',w})' = M_{E,w}$$

as sets with the gauge $\| \cdot \|_{M_{E,w}}$ equivalent to the norm $\| \cdot \|_{(\Lambda_{E',w})'}$.

Spaces $Q_{E,w}$

Definition

We denote by $Q_{E,w}$ the set of elements of $L^0 = L^0(I)$ which are submajorized by an element of $M_{E,w}$. For $f \in Q_{E,w}$ we set

$$\|f\|_{Q_{E,w}} = \inf\{\|g\|_{M_{E,w}} : f \prec g\}.$$

$W < \infty$, **Marcinkiewicz function space**

$$M_w = \left\{ f \in L^0 : \|f\|_{M_w} = \sup_{t \in I} \frac{\int_0^t f^*}{W(t)} < \infty \right\},$$

and the **space** $L_1 + M_w$ is the set of all functions $f \in L^0$ such that

$$\|f\|_{L_1+M_w} = \inf\{\|h\|_1 + \|g\|_{M_w} : \ f = h + g, \ h \in L_1, \ g \in M_w\}.$$

$(M_w, \|\cdot\|_{M_w})$, $(L_1 + M_w, \|\cdot\|_{L_1+M_w})$ are fully symmetric

Theorem

Let w be a weight function such that $W < \infty$ on I.

(i) *The class $Q_{E,w}$ is a solid linear subspace of $L_1 + M_w$ such that*

$$\|f\|_{L_1+M_w} \leq C\|f\|_{Q_{E,w}} \quad \text{with} \quad C \leq (1 \wedge b)/\phi_E(1 \wedge b).$$

(ii) *The functional $\|\cdot\|_{Q_{E,w}}$ is a norm on $Q_{E,w}$.*

(iii) *$(Q_{E,w}, \|\cdot\|_{Q_{E,w}})$ is the smallest fully symmetric Banach function space containing the class $M_{E,w}$.*

(iv) *We have $(Q_{E,w})' = \Lambda_{E',w}$ with equality of norms.*

Let E have the Fatou property. Then $E = E''$. Hence

$$\Lambda_{E,w} = \Lambda_{E'',w} = (Q_{E',w})'$$

and

$$(\Lambda_{E,w})' = ((Q_{E',w})')' = (Q_{E',w})''$$

If **we know that $Q_{E,w}$ has the Fatou property** whenever E has it then

$$Q_{E',w} = (Q_{E',w})''$$

and **the Köthe dual space is as follows**

$$(\Lambda_{E,w})' = Q_{E',w}.$$

It seems to be difficult to show this without the knowledge of the **minimizer** g in the formula for the norm

$$\|f\|_{Q_{E,w}} = \inf\{\|g\|_{M_{E,w}} : f \prec g\}.$$

This minimizer will appear to be a level function f^0 of f.

Link with Halperin's level functions

$W < \infty$ on I. For $f = f^*$ locally integrable on I, define after Halperin (1953) for $0 \le \alpha < \beta < \infty$, $\alpha, \beta \in I = (0, a)$, $a \le \infty$,

$$W(\alpha,\beta) = \int_\alpha^\beta w dm, \quad F(\alpha,\beta) = \int_\alpha^\beta f\, dm, \quad R(\alpha,\beta) = \frac{F(\alpha,\beta)}{W(\alpha,\beta)},$$

and for $\beta = \infty$,

$$R(\alpha,\beta) = R(\alpha,\infty) = \limsup_{t\to\infty} R(\alpha,t).$$

Then $(\alpha,\beta) \subset I$ is called a **level interval (resp. degenerate level interval) of** f **with respect to** w if $\beta < \infty$ (resp. $\beta = \infty$) and for each $t \in (\alpha,\beta)$,

$$R(\alpha,t) \le R(\alpha,\beta) \text{ and } 0 < R(\alpha,\beta).$$

If a level interval is not contained in any larger level interval, then it is called **maximal level interval** (m.l.i.). The m.l.i. of f are pairwise disjoint and unique, there is at most countable number of m.l.i.

I. Halperin, 1953. Let $f \in L^0$ be non-negative, decreasing and locally integrable on I. Then the **level function** f^0 of f with respect to w is defined as

$$f^0(t) = \begin{cases} R(\alpha, \beta)\, w(t) & \text{if } t \text{ belongs to some maximal level interval } (\alpha, \beta), \\ f(t) & \text{otherwise.} \end{cases}$$

For a general $f \in L^0$, $0 \le \alpha < \beta < \infty$, $\alpha, \beta \in I$, we define

$$f^0 = (f^*)^0, \quad F(\alpha, \beta) = \int_\alpha^\beta f^*\, dm, \quad \text{and} \quad F(t) = \int_0^t f^*\, dm, \quad t \in I.$$

Fact (Properties of level functions)

Let $f \in L_1 + L_\infty$ and w be a decreasing locally integrable weight function on I.

(i) f^0/w *is decreasing. Consequently in view of* w *being decreasing,* f^0 *is decreasing as well.*

(ii) $f \prec f^0$. *Moreover if* x *does not belong to a m.l.i.,* $\int_0^x f^0\,dm = \int_0^x f^*\,dm$, *and so if* I *is finite,* $\int_I f^0\,dm = \int_I f^*\,dm$.

(iii) *If* $f \prec g$ *then* $f^0 \prec g^0$.

Lemma

If $f \in L_1 + M_w$ then $f^0 \in L_1 + M_w$, and $\|f\|_{L_1+M_w} = \|f^0\|_{L_1+M_w}$.

Proof.

Assume $\|f\|_{L_1+M_w} < 1$. We have $f = g + h$ with some $g \in L_1$, $h \in M_w$ such that $\|g\|_1 + \|h\|_{M_w} < 1$. Then $f^* \prec g^* + h^* \prec g^* + \|h\|_{M_w} w$. It follows that $f^0 \prec (g^* + \|h\|_{M_w} w)^0$. It is easy to see that $g^* + \|h\|_{M_w} w$ and g^* have the same m.l.i. and that $(g^* + Cw)^0 = g^0 + Cw$, $C = \|h\|_{M_w}$. Then

$$\|f^0\|_{L_1+M_w} \leq \|g^0 + Cw\|_{L_1+M_w} \leq \|g^0\|_1 + \|h\|_{M_w} \|w\|_{M_w} = \|g\|_1 + \|h\|_{M_w} < 1.$$

This shows that $\|f^0\|_{L_1+M_w} \leq \|f\|_{L_1+M_w}$ for every $f \in L_1 + M_w$. The converse inequality follows from $f \prec f^0$. □

Theorem

A function $f \in L_1 + M_w$ belongs to $Q_{E,w}$ if and only if its level function f^0 relative to w belongs to $M_{E,w}$, and then

$$\|f\|_{Q_{E,w}} = \|f^0\|_{M_{E,w}}.$$

Theorem

If E has the Fatou property then so has $Q_{E,w}$, and moreover $(\Lambda_{E',w})' = Q_{E,w}$ with equal norms.

If $Q_{E,w}$ has the Fatou property, then we have $\Lambda_{E',w} = (Q_{E,w})'$ with equal norms. It follows that $(\Lambda_{E',w})' = (Q_{E,w})'' = Q_{E,w}$ with equal norms.
It remains to prove that $Q_{E,w}$ has the Fatou property when E has the property.

Despite that the space E_w is symmetric with respect to the measure $\omega = wdm$, the space $(E_w)'$ will always denote its Köthe dual computed with respect to the Lebesgue measure m as it is done below.

Lemma

For any $f \in (E_w)'$ we have $\|f\|_{(E_w)'} = \left\|\frac{f}{w}\right\|_{(E')_w}$. Moreover $(E_w)'' = (E'')_w$ with equality of norms.

$$
\begin{aligned}
\|f\|_{(E_w)'} &= \sup\left\{\int_I |f| g\,dm : \|g\|_{E_w} \le 1\right\} = \sup\left\{\int_I \frac{|f|}{w} g\,wdm : \|g\|_{E_w} \le 1\right\} \\
&= \sup\left\{\int_J \left(\frac{|f|}{w} \circ W^{-1}\right) \cdot (g \circ W^{-1})\,dm : \|g \circ W^{-1}\|_E \le 1\right\} \\
&= \sup\left\{\int_J \left(\frac{|f|}{w} \circ W^{-1}\right) \cdot h\,dm : \|h\|_E \le 1\right\} \\
&= \left\|\frac{|f|}{w} \circ W^{-1}\right\|_{E'} = \left\|\frac{f}{w}\right\|_{(E')_w}.
\end{aligned}
$$

This proves the first part of the Lemma. Using this part once for E, then for E' we get:

$$
\begin{aligned}
\|f\|_{(E_w)''} &= \sup\left\{\int_I |f| h\, dm : \|h\|_{(E_w)'} \leq 1, \right\} \\
&= \sup\left\{\int_I |f| \frac{h}{w}\, w dm : \left\|\frac{h}{w}\right\|_{(E')_w} \leq 1\right\} \\
&= \sup\left\{\int_I (|f| w) g\, dm : \|g\|_{(E')_w} \leq 1\right\} = \left\|\frac{|f| w}{w}\right\|_{(E'')_w} \\
&= \|f\|_{(E'')_w}.
\end{aligned}
$$

which proves the second part.

Lemma

It holds $(\Lambda_{E,w})'' = \Lambda_{E'',w}$ *with equal norms.*

We use the fact that if F is a Banach function space, then $f \geq 0$ belongs to F'' with $\|f\|_{F''} \leq 1$ if and only if there exists a sequence $0 \leq f_n \uparrow f$, with $f_n \in F$, $\|f_n\|_F \leq 1$ for all $n \in \mathbb{N}$.

Let $\|f\|_{(\Lambda_{E,w})''} \leq 1$, let $0 \leq f_n \uparrow f$ with $\|f_n\|_{\Lambda_{E,w}} \leq 1$. Then $f_n^* \uparrow f^*$, and $f_n^* \in E_w$, $\|f_n^*\|_{E_w} \leq 1$. Hence $f^* \in (E_w)''$ with $\|f\|_{(E_w)''} \leq 1$. However by Lemma 20, $(E_w)'' = (E'')_w$ and so $f \in \Lambda_{E'',w}$ with $\|f\|_{\Lambda_{E'',w}} \leq 1$.

Conversely, let $f \in \Lambda_{E'',w}$ with norm ≤ 1. Then $f^* \in (E'')_w$ with $\|f\|_{\Lambda_{E'',w}} = \|f^*\|_{(E'')_w} \leq 1$. Since $(E'')_w = (E_w)''$, there exists $0 \leq g_n \uparrow f^* \in E_w$, with $\|g_n\|_{E_w} \leq 1$. Then

$$g_n^{*,w} = (g_n \circ W^{-1})^* \uparrow f^* \circ W^{-1}.$$

Setting $h_n = g_n^{*,w} \circ W$, we have that $h_n \geq 0$ and these functions are decreasing on I. Clearly $h_n \uparrow f^*$ and $h_n^{*,w} = g_n^{*,w}$, so $\|h_n\|_{\Lambda_{E,w}} = \|h_n\|_{E_w} = \|h_n^{*,w}\|_E = \|g_n^{*,w}\|_E = \|g_n\|_{E_w} \leq 1$. Therefore $f^* \in (\Lambda_{E,w})''$ with $\|f\|_{(\Lambda_{E,w})''} = \|f^*\|_{(\Lambda_{E,w})''} \leq 1$, which shows the desired equality of spaces and norms.

In the next result we **do not** assume the Fatou property for E.

Corollary

Let w be a decreasing positive weight on I and $W < \infty$. We have $(\Lambda_{E,w})' = Q_{E',w}$ with equal norms.

Proof.

By general theory of Banach function lattices

$$(\Lambda_{E,w})' = (\Lambda_{E,w})'''.$$

Since E' has the Fatou then

$$Q_{E',w} = (\Lambda_{E'',w})',$$

$$Q_{E',w} = (\Lambda_{E'',w})' = [(\Lambda_{E,w})'']' = (\Lambda_{E,w})''' = (\Lambda_{E,w})'.$$

□

.

Theorem

Let w be a decreasing positive weight on I and $W < \infty$. For $f \in L^0$ we have

$$\sup\left\{\int_I |fg| : g \in \Lambda_{E,w}, \|g\|_{\Lambda_{E,w}} \leq 1\right\} = \begin{cases} \|f^0\|_{M_{E',w}} & \text{if } f^0 \in M_{E',w}, \\ \infty & \text{otherwise.} \end{cases}$$

Consequently $\|f\|_{(\Lambda_{E,w})'} = \|f^0\|_{M_{E',w}} = \|f\|_{Q_{E',w}}$ *for every* $f \in (\Lambda_{E,w})'$.

Spaces $P_{E,w}$

We assume in this chapter that $W < \infty$ on I.

Definition

We denote by $P_{E,w}$ the union of the classes $M_{E,v}$, where v is a positive decreasing weight submajorized by w on I. This set is equipped with the gauge

$$\|f\|_{P_{E,w}} = \inf\left\{\|f\|_{M_{E,v}} : v > 0, v\downarrow, v \prec w\right\}.$$

Our goal is to show that $\|\cdot\|_{P_{E,w}}$ is a symmetric norm, and in fact the $P_{E,w} = Q_{E,w}$ as sets and $\|\cdot\|_{P_{E,w}} = \|\cdot\|_{Q_{E,w}}$. We will also obtain another duality formula for $\Lambda_{E,w}$.

From the next lemma it follows that the gauge on $P_{E,w}$ is faithful.

Lemma

We have $M_{E,w} \subset P_{E,w} \subset M_{E,\tilde{w}}$, where $\tilde{w}(t) := \frac{W(t)}{t}$, $t \in I$, and these inclusions are gauge-decreasing.

Proof.

The first inclusion and the corresponding gauge inequality are clear. Conversely for each $v \prec w$ we have $tv(t) \leq V(t) \leq W(t)$, and thus $v(t) \leq \tilde{w}(t)$, $t \in I$, and in view of Lemma 10, $M_{E,v} \subset M_{E,\tilde{w}}$, with $\|\cdot\|_{M_{E,\tilde{w}}} \leq \|\cdot\|_{M_{E,v}}$. Taking the infima with respect to $v \prec w$ we obtain $P_{E,w} \subset M_{E,\tilde{w}}$ with $\|\cdot\|_{M_{E,\tilde{w}}} \leq \|\cdot\|_{P_{E,w}}$. □

Lemma

If $v \prec w$ and $h \in E_w$ is decreasing then $h \in E_v$ and $\|h\|_{E_v} \le \|h\|_{E_w}$.

Proof.

By Hardy's Lemma [1, Proposition 3.6, p. 56] since $(h-\lambda)_+$ is decreasing and $v \prec w$, for every $\lambda > 0$ we have

$$\int_I (h-\lambda)_+ \, v dm \le \int_I (h-\lambda)_+ \, w dm.$$

Then in view of identity (5) for any $x \in J$,

$$\int_0^x h^{*,v} \, dm = \inf_{\lambda>0} \left[\int_I (h-\lambda)_+ \, v dm + \lambda x \right] \frac{1}{2}$$
$$\le \inf_{\lambda>0} \left[\int_I (h-\lambda)_+ \, w dm + \lambda x \right] = \frac{1}{2} \int_0^x h^{*,w} \, dm,$$

and so $h^{*,v} \prec h^{*,w}$. Thus since E is fully symmetric and $h^{*,w} \in E$ we have that $h^{*,v} \in E$ and so $h \in E_v$. Moreover $\|h\|_{E_v} = \|h^{*,v}\|_E \le \|h^{*,w}\|_E = \|h\|_{E_w}$. □

Theorem

We have $(P_{E,w})' = \Lambda_{E',w}$ *with equal norms.*

Theorem

$Q_{E,w} \subset P_{E,w}$ *and the inclusion is gauge decreasing.*

Proof.

By Theorem 18 we have $\|f\|_{Q_{E,w}} = \|f^0\|_{M_{E,w}}$. Clearly $\frac{f^*}{w^f} = \frac{f^0}{w}$. By Fact 16(i) the latter function is decreasing. Hence by Lemma 26 and Theorem 18 we get

$$\|f\|_{M_{E,w^f}} = \left\|\frac{f^*}{w^f}\right\|_{E_{w^f}} \leq \left\|\frac{f^*}{w^f}\right\|_{E_w} = \left\|\frac{f^0}{w}\right\|_{E_w} = \|f^0\|_{M_{E,w}} = \|f\|_{Q_{E,w}},$$

and a fortiori $\|f\|_{P_{E,w}} \leq \|f\|_{Q_{E,w}}$. □

Corollary

If E has the Fatou property then $P_{E,w} = Q_{E,w}$ isometrically, that is $\|f\|_{P_{E,w}} = \|f\|_{Q_{E,w}}$ for every $f \in P_{E,w}$. Consequently the class $P_{E,w}$ is a fully symmetric Banach function space having all properties discussed earlier.

Proof.

We have

$$Q_{E,w} \subset P_{E,w} \subset (P_{E,w})'' = (\Lambda_{E',w})' = Q_{E'',w},$$

and these inclusions are gauge decreasing.
If E has the Fatou property then $E'' = E$ with equal norms, hence these inclusions are norm preserving equalities. □

Since E' has the Fatou property we get the following result.

Corollary

For any fully symmetric Banach function space E, we have $(\Lambda_{E,w})' = P_{E',w}$ isometrically.

Theorem

Let $w : I = (0,a) \to (0,\infty)$ *be non-increasing, E fully symmetric function space on* $J = (0,b), b = W(a) = \int_0^a w$*. Then*

$$(\Lambda_{E,w})' = Q_{E',w} = P_{E',w},$$

and

$$\|f\|_{(\Lambda_{E,w})'} = \|f^0\|_{M_{E',w}} = \|f\|_{Q_{E',w}} = \|f\|_{P_{E',w}}.$$

Applications to modular and Orlicz-Lorentz spaces

1.
$E = \Lambda_{E_1,w_1}$, where w_1 is a decreasing weight on J,
$J_1 = (0, W_1(b)) = (0, W_1 \circ W(a))$ and E_1 is a fully symmetric Banach function space on J_1. Then for $f \in L_1 + L_\infty(I)$ we have

$$f \in \Lambda_{E,w} \iff f^* \circ W^{-1} \in \Lambda_{E_1,w_1} \iff f^* \circ W^{-1} \circ W_1^{-1} \in E.$$

Setting $U = W_1 \circ W$ and $u = (w_1 \circ W)w$, u is a decreasing weight on I with $U(t) = \int_0^t u\, dm$, $t \in I$, and

$$\Lambda_{E,w} = \Lambda_{E_1,u}$$

with equal norms.

2. Let X be a real vector space. For an extended real valued functional $\rho : X \to [0, \infty]$ consider the following conditions.

(i) $\rho(0) = 0$ and if $x \in X$ and $\rho(tx) = 0$ for every $t \geq 0$ then $x = 0$.

(ii) $\rho(-x) = \rho(x)$.

(iii) ρ is , that is for every $t_1, t_2 \geq 0$, $t_1 + t_2 = 1$, and every $x_1, x_2 \in X$, we have $\rho(t_1 x_1 + t_2 x_2) \leq t_1 \rho(x_1) + t_2 \rho(x_2)$.

(iii') For every $x \in X$, the extended real valued function $t \to \rho(tx)$ is convex.

If ρ satisfies all conditions $(\mathrm{i}) - (\mathrm{iii})$ then ρ is called a **modular**. If ρ fulfills $(\mathrm{i}), (\mathrm{ii}), (\mathrm{iii}')$ then ρ will be called a **pseudo-modular**. Given (pseudo-) modular ρ, **the modular space** X_ρ consists of all $x \in X$ such that $\rho(tx) < \infty$ for some $t > 0$. It is easy to check that X_ρ is a vector space. There are **two gauges on** X_ρ classically associated with the (pseudo-) modular ρ. These gauges are norms if ρ is a modular.

If ρ is a modular then we define the norms as follows.

The **Luxemburg** (or **second Nakano**) norm is the Minkowski functional of the convex set $U = \{x \in E : \rho(x) \leq 1\}$, thus

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho(x/\lambda) \leq 1\}, \tag{5}$$

the **Orlicz** (or **first Nakano**) norm is given by **Amemiya's formula**

$$\|x\|_\rho^0 = \inf_{\lambda>0} \frac{1+\rho(\lambda x)}{\lambda} = \inf_{t>0} \left(t + t\rho\left(\frac{x}{t}\right)\right). \tag{6}$$

There is **another expression of the Luxemburg norm**, similar to Amemiya's formula. In fact we have

$$\|x\|_\rho = \inf_{\lambda>0} \frac{1 \vee \rho(\lambda x)}{\lambda} = \inf_{t>0} \left(t \vee t\rho\left(\frac{x}{t}\right)\right). \tag{7}$$

Lemma

Let $\psi : [0, \infty) \to [0, \infty)$ *be a convex increasing function. If* $f, g \in L_1 + L_\infty(\Omega)$ *with* $f \prec g$ *then* $\psi(f) \prec \psi(g)$.

Theorem

Let φ be an Orlicz function and w be a weight function on $I = (0, a)$, $a \le \infty$, such that $W < \infty$ on I. Then

$$\mathrm{Q}(f) = \inf\left\{\int_I \varphi\left(\frac{g^*}{w}\right) w\,dm : f \prec g\right\} = \mathrm{P}(f) = \inf\left\{\int_I \varphi\left(\frac{f^*}{v}\right) v\,dm : v \prec w,\right.$$

$$= \int_I \varphi\left(\frac{f^0}{w}\right) w\,dm.$$

For $f \in \mathcal{M}_{\varphi,w}$ we have

$$\|f\|_{\mathcal{M}} = \inf\{\|g\|_{\mathrm{M}};\ f \prec g\} = \inf\{\|f\|_{\mathrm{M}_v} :\ v \prec w, v > 0,\ v \downarrow\},$$

$$\|f\|_{\mathcal{M}}^0 = \inf\{\|g\|_{\mathrm{M}}^0;\ f \prec g\} = \inf\{\|f\|_{\mathrm{M}_v}^0 :\ v \prec w, v > 0,\ v \downarrow\},$$

where

$$\mathrm{M}(f) = \int_I \phi\left(\frac{f^*}{w}\right) w\,dm \quad \textit{and} \quad \mathrm{M}_v(f) = \int_I \varphi\left(\frac{f^*}{v}\right) v\,dm,$$

and $\|\cdot\|_{\mathrm{M}}, \|\cdot\|_{\mathrm{M}_v}$ are Luxemburg, and $\|\cdot\|_M^0, \|\cdot\|_{M_v}^0$ are Amemiya gauges.

Examples of $M_{E,w}$ and $Q_{E,w}$ spaces

(1) *If* $E = L_1$*, then* $(M_{L_1,w}, \|\cdot\|_{M_{L_1,w}}) = (Q_{L_1,w}, \|\cdot\|_{Q_{L_1,w}}) = (L_1, \|\cdot\|_1)$.

Proof.

Clearly $E_w = (L_1)_w$ is a weighted L_1 space. We also have

$$f \in M_{L_1,w} \iff \frac{f^*}{w} \in (L_1)_w \iff \int_I \frac{f^*}{w} w \, dm < \infty$$
$$\iff \int_I f^* \, dm < \infty \iff f \in L_1.$$

Hence $M_{L_1,w} = L_1$ with the same norms. It follows that $Q_{L_1,w} = L_1$, also with the same norms. □

(2) *If* $E = L_\infty$ *then*

$$M_{L_\infty,w} = \{f : \|f\|_{M_{L_\infty,w}} = \inf\{C : f^* \leq Cw\} < \infty\},$$

$$(Q_{L_\infty,w}, \|\cdot\|_{Q_{L_\infty,w}}) = (M_W, \|\cdot\|_{M_W}).$$

(3) *If* $E = L_1 \cap L_\infty$ *then*

$$(M_{L_1 \cap L_\infty,w}, \|\cdot\|_{M_{L_1 \cap L_\infty,w}}) = (L_1 \cap M_{L_\infty,w}, \|\cdot\|_{L_1 \cap M_{L_\infty,w}})$$

$$(Q_{L_1 \cap L_\infty,w}, \|\cdot\|_{Q_{L_1 \cap L_\infty,w}} = (L_1 \cap M_W, \|\cdot\|_{L_1 \cap M_W}).$$

(4) *If* $E = L_1 + L_\infty$ *then*

$$(M_{L_1 + L_\infty,w}, \|\cdot\|_{M_{L_1 + L_\infty,w}}) = (L_1 + M_{L_\infty,w}, \|\cdot\|_{L_1 + M_{L_\infty,w}}),$$

$$(Q_{L_1 + L_\infty,w}, \|\cdot\|_{Q_{L_1 + L_\infty,w}}) = (L_1 + M_W, \|\cdot\|_{L_1 + M_W}).$$

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