A remark on smooth images of Banach spaces

Petr Hájek, Michal Johanis
Theorem 1

Let $Y$ be a separable Banach space. Then there exists a bounded linear operator from $\ell_1$ onto $Y$. 
Smooth surjections

Theorem 1

Let $Y$ be a separable Banach space. Then there exists a bounded linear operator from $\ell_1$ onto $Y$.

Theorem 2 (Sean Michael Bates, 1997)

Let $X$ be an infinite-dimensional Banach space and $Y$ a separable Banach space. Then there exists a $C^1$-smooth Lipschitz mapping from $X$ onto $Y$. 
Smooth surjections

**Theorem 1**

Let $Y$ be a separable Banach space. Then there exists a bounded linear operator from $\ell_1$ onto $Y$.

**Theorem 2 (Sean Michael Bates, 1997)**

Let $X$ be an infinite-dimensional Banach space and $Y$ a separable Banach space. Then there exists a $C^1$-smooth Lipschitz mapping from $X$ onto $Y$.

**Theorem 3 (Sean Michael Bates, 1997)**

If a Banach space $X$ has property $B$, then for any separable Banach space $Y$ there exists a $C^\infty$-smooth mapping from $X$ onto $Y$. 

Petr Hájek, Michal Johanis

A remark on smooth images of Banach spaces
Theorem 4 (Petr Hájek, 1999)

Let $X$ be a Banach space such that there exists a non-compact operator $T \in \mathcal{L}(X; \ell_p)$, $1 \leq p < \infty$. Then for any separable Banach space $Y$ there exists a $\lceil p \rceil$-homogeneous polynomial surjection $P : X \to Y$. 

Note that this is also a direct generalisation of Theorem 1.

P. Hájek (1998): There is no $C^2$-smooth surjection $f : c_0 \to \ell_2$. 

Petr Hájek, Michal Johanis
A remark on smooth images of Banach spaces
Theorem 4 (Petr Hájek, 1999)

Let $X$ be a Banach space such that there exists a non-compact operator $T \in \mathcal{L}(X; \ell_p)$, $1 \leq p < \infty$. Then for any separable Banach space $Y$ there exists a $\lceil p \rceil$-homogeneous polynomial surjection $P : X \to Y$.

Note that this is also a direct generalisation of Theorem 1.
Theorem 4 (Petr Hájek, 1999)

Let $X$ be a Banach space such that there exists a non-compact operator $T \in \mathcal{L}(X; \ell_p)$, $1 \leq p < \infty$. Then for any separable Banach space $Y$ there exists a $\lceil p \rceil$-homogeneous polynomial surjection $P : X \to Y$.

Note that this is also a direct generalisation of Theorem 1.

P. Hájek (1998):
There is no $C^2$-smooth surjection $f : c_0 \to \ell_2$. 
The situation depends on axioms of set theory:
The situation depends on axioms of set theory:

- **CH**: There is a $C^\infty$-smooth surjection from $c_0(\omega_1)$ onto $\ell_2$. (easy)
The situation depends on axioms of set theory:

- **CH**: There is a $C^\infty$-smooth surjection from $c_0(\omega_1)$ onto $\ell_2$. (easy)

- **MA$_{\omega_1}$**: There is no $C^2$-smooth surjection from $c_0(\omega_1)$ onto $\ell_2$ (follows from the results of P. Hájek (1998), formulated in [Guirao, Hájek, Montesinos, 2010]).
Theorem 5 (Robert G. Bartle and Lawrence M. Graves, 1952)

Let $X$, $Y$ be Banach spaces and let $T \in \mathcal{L}(X; Y)$ be onto. Then there is a subspace $Z \subset X$ with $\text{dens } Z = \text{dens } Y$ such that $T|_Z$ is still surjective.
Smooth surjections: the non-separable case

Theorem 5 (Robert G. Bartle and Lawrence M. Graves, 1952)

Let $X$, $Y$ be Banach spaces and let $T \in \mathcal{L}(X; Y)$ be onto. Then there is a subspace $Z \subset X$ with $\text{dens } Z = \text{dens } Y$ such that $T \upharpoonright Z$ is still surjective.

Theorem 6 (Richard M. Aron, Jesús A. Jaramillo, and Enrico Le Donne, 2017)

Let $X$, $Y$ be Banach spaces and let $f \in C^1(X; Y)$ be onto and such that the set of critical values of $f$ has cardinality at most $\text{dens } Y$. Then there is a subspace $Z \subset X$ with $\text{dens } Z = \text{dens } Y$ such that $f \upharpoonright Z$ is still surjective.
Theorem 7 (Richard M. Aron, Jesús A. Jaramillo, and Thomas Ransford, 2013)

Let \( \Gamma \) be a set of cardinality at least continuum and suppose there exists a bounded linear operator \( T : X \to c_0(\Gamma) \) such that \( T(X) \) contains the canonical basis of \( c_0(\Gamma) \). Then for any separable Banach space \( Y \) of dimension at least two there exists a \( C^\infty \)-smooth surjective mapping \( f : X \to Y \) such that the restriction of \( f \) onto any separable subspace of \( X \) fails to be surjective.
Let $\Gamma$ be a set of cardinality at least continuum and suppose there exists a bounded linear operator $T : X \to c_0(\Gamma)$ such that $T(X)$ contains the canonical basis of $c_0(\Gamma)$. Then for any separable Banach space $Y$ of dimension at least two there exists a $C^\infty$-smooth surjective mapping $f : X \to Y$ such that the restriction of $f$ onto any separable subspace of $X$ fails to be surjective.

The theorem holds in particular for $X = \ell_p(\Gamma)$, $\text{card} \Gamma \geq c$.

**Question:** Does it hold also for $X = \ell_p(\Gamma)$, $\text{card} \Gamma = \omega_1$? Does it involve axioms of set theory?
Theorem 8

Let $X$ be a Banach space for which there is $T \in \mathcal{L}(X; \ell_p(\Gamma))$ for some uncountable $\Gamma$ and $1 \leq p < \infty$ such that $T(B_X)$ contains the canonical basis of $\ell_p(\Gamma)$.

Then for any separable Banach space $Y$ with $\dim Y \geq 2$ there is $f \in C^\infty(X; Y)$ such that $f(X) = Y$ but $f(Z) \neq Y$ for any separable subset $Z \subset X$. 
Theorem 8

Let $X$ be a Banach space for which there is $T \in \mathcal{L}(X; \ell_p(\Gamma))$ for some uncountable $\Gamma$ and $1 \leq p < \infty$ such that $T(B_X)$ contains the canonical basis of $\ell_p(\Gamma)$. 
(This holds in particular if $X$ is a non-separable super-reflexive space.) Then for any separable Banach space $Y$ with $\dim Y \geq 2$ there is $f \in C^\infty(X; Y)$ such that $f(X) = Y$ but $f(Z) \neq Y$ for any separable subset $Z \subset X$. 
Theorem 9 (Felix Hausdorff, 1936)

Every uncountable Polish space is a union of an increasing $\omega_1$-sequence of $G_\delta$ sets.
Theorem 9 (Felix Hausdorff, 1936)

Every uncountable Polish space is a union of an increasing $\omega_1$-sequence of $G_\delta$ sets.

Theorem 10

Let $X$ be an infinite-dimensional Banach space that admits a $C^k$-smooth bump, $k \in \mathbb{N} \cup \{\infty\}$, with each derivative bounded on $X$. Let $Y$ be a separable Banach space, let $C \subset Y$ be convex, $y_1 \in C$, and $C \subset A \subset \overline{C}$ an analytic set. Then there is $f \in C^k(X; Y)$ with $\text{supp}_o f \subset B_X$ such that $f(X) = [0, y_1] \cup A$. 
Proposition 11

Let \((X, \rho)\) be a separable metric space, \(U \subset X\), \(A \subset \overline{U}\) a non-empty Suslin set. Then there is an \(\omega\)-branching tree \(T\) of height \(\omega\) with a least element and a family \(\{x_t\}_{t \in T} \subset U\) such that

\[
A = \left\{ \lim_{n \to \infty} x_{b_n}; \ b \in B(T) \right\}.
\]
Proposition 11

Let \((X, \rho)\) be a separable metric space, \(U \subset X\), \(A \subset \overline{U}\) a non-empty Suslin set, and let \(\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, +\infty)\). Then there is an \(\omega\)-branching tree \(T\) of height \(\omega\) with a least element and a family \(\{x_t\}_{t \in T} \subset U\) such that
\[
A = \left\{ \lim_{n \to \infty} x_{b_n}; \ b \in \mathcal{B}(T) \right\} \quad \text{and} \quad \rho(x_u, x_t) < \varepsilon_n \quad \text{for each} \quad u \in t^+, \ t \in T_n, \ n \in \mathbb{N}.
\]
Let $X$ be a Banach space for which there is $T \in \mathcal{L}(X; \ell_p(\Gamma))$ for some infinite $\Gamma$ and $1 \leq p < \infty$ such that $T(B_X)$ contains the canonical basis of $\ell_p(\Gamma)$. Then for every Banach space $Y$ of density at most $\text{card } \Gamma$ there exists a $\lceil p \rceil$-homogeneous polynomial surjection $P : X \to Y$. 
Theorem 13

Let $X$ be a Banach space, $\mu > \omega$ a regular cardinal, $\Gamma$ a set, $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Consider the following statements:

(i) $X$ is WCG with $\text{dens } X \geq \mu$ and $X^*$ is $w^*$-$\ell_q(\Gamma)$-generated.

(ii) $X$ contains a non-zero weakly null net $\{x_\alpha\}_{\alpha \in [0,\mu)}$ and there is $T \in \mathcal{L}(X; \ell_p(\Gamma))$ such that $\text{dens ker } T < \mu$.

(iii) There is $T \in \mathcal{L}(X; \ell_p([0, \mu)))$ such that $\{e_\gamma\}_{\gamma \in [0, \mu)} \subset T(B_X)$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).
Definition

We say that a Banach space $X$ has property $B$ if $X^*$ contains a normalised weakly null hereditarily Banach-Saks sequence.
We say that a Banach space $X$ has property $B$ if $X^*$ contains a normalised weakly null hereditarily Banach-Saks sequence.

Note that a Banach space contains a normalised weakly null sequence if and only if it is not a Schur space. In particular, if $X$ is an infinite-dimensional Banach space such that $X^*$ has the Banach-Saks property (or more generally if $X^*$ is not a Schur space and has the weak Banach-Saks property), then $X$ has property $B$. 
We say that a Banach space $X$ has property $B$ if $X^*$ contains a normalised weakly null hereditarily Banach-Saks sequence.

Note that a Banach space contains a normalised weakly null sequence if and only if it is not a Schur space. In particular, if $X$ is an infinite-dimensional Banach space such that $X^*$ has the Banach-Saks property (or more generally if $X^*$ is not a Schur space and has the weak Banach-Saks property), then $X$ has property $B$.

**Proposition 14**

Let $X$ be a Banach space with a sub-symmetric Schauder basis. Then $X$ has property $B$ if and only if $X$ is not isomorphic to $c_0$. 
A tree is a partially ordered set \((T, \preceq)\) with the property that for every \(t \in T\) the subset \(\{s \in T; s \preceq t\}\) is well-ordered.

For \(t \in T\) we denote by \(t^+\) the set of all immediate successors of \(t\), i.e.
\[
t^+ = \{u \in T; s \prec u \text{ if and only if } s \preceq t\}.
\]
The height of \(t \in T\) is a unique ordinal \(\text{ht}(t)\) with the same order type as \(\{s \in T; s \prec t\}\). The height of the tree \(T\) is defined by \(\sup \{\text{ht}(t) + 1; t \in T\}\).

A branch of \(T\) is a maximal linearly ordered subset and we denote by \(B(T)\) the set of all branches of \(T\). For an ordinal \(\alpha\) we denote by \(T_\alpha = \{t \in T; \text{ht}(t) = \alpha\}\) the \(\alpha\)th level of the tree \(T\).

For a branch \(b \in B(T)\) we denote \(b_\alpha = b \cap T_\alpha\). Let \(\mu\) be a cardinal. We say that \(T\) is \(\mu\)-branching if \(\text{card } T_0 \leq \mu\) and \(\text{card } t^+ < \mu\) for each \(t \in T\).
Let $\mu$ be a cardinal. We say that a subset $S$ of a topological space $X$ is $\mu$-Suslin in $X$ if there is a $\mu$-branching tree $T$ of height $\omega$ and closed sets $F_t \subset X$, $t \in T$ such that $S = \bigcup_{b \in B(T)} \bigcap_{n=1}^{\infty} F_{b_n}$. We remark that $\omega$-Suslin sets are called simply Suslin in the classical terminology and that a classical result states that in Polish spaces Suslin sets (our $\omega$-Suslin sets) are precisely the analytic sets.